

On a bipolar foundation of the outranking concept

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ABSTRACT In this paper we introduce a bipolarly extended veto principle which allows us to extend the definition of the classic outranking relation in such a way that the identity between the asymmetric part and bipolar codual of the latter outranking relation is given.

KEYWORDS. Multiple criteria decision aid; Outranking approach; Bipolar veto principle.

1. INTRODUCTION

Recently, Pirlot and Bouyssou (2009) have reported that a *strict* (asymmetric) outranking relation defined similarly to the classic outranking relation (Roy and Bouyssou, 1993) is in general not identical to its codual relation, that is the converse of its negation. Indeed, from value-based orderings, we are used to think that a decision alternative x is considered *strictly better* than a decision alternative y , when it is *not true* that y is *at least as good as* x . Thereby, we genuinely expect the '*strictly better*' relation to be asymmetric. This will however only be the case if the corresponding '*at least as good as*' relation is complete, a fact which is generally not verified when dealing with classic outranking relations. This hiatus is problematic as the asymmetric part of an outranking relation is commonly identified in the decision aid practice as representing in fact its codual relation.

In this contribution we explore this problem in the context of our bipolar credibility calculus (Bisdorff, 2000, 2002, 2006). Characteristic functions may here denote the potential truth or not of a preferential statement with the help of three states : *more true than false*, *more false than true*, or *logically indeterminate*. Most important to notice is that in this setting the logical negation operation can no longer simply be identified with a standard set complementation. Contrary to classic logics, affirmation, as well as refutation of a preferential statement have therefore both to be based on explicit, not necessarily complementary, arguments.

In a first section, following the hint of Pirlot and Bouyssou, we illustrate formally this unsound hiatus between the asymmetric part and the codual in the case of the classic outranking concept. In a second we introduce a bipolarly extended veto principle which allows us to extend the definition of the classic outranking relation in such a way that the identity between the asymmetric part and the bipolar codual of the outranking relation is indeed given.

2. THE CLASSIC OUTRANKING CONCEPT

Let $A = \{x, y, z, \dots\}$ be a finite set of potential decision alternatives and let $F = \{1, \dots, n\}$ be a coherent finite family of $n > 1$ criteria (Roy and Bouysoou, 1993). The alternatives are evaluated on each criterion i in F on a real performance scale $[0; M_i]$ supporting coherent indifference (q_i) and preference (p_i) discrimination thresholds such that $0 \leq q_i < p_i \leq M_i$ (Roy and Bouysoou, 1993). The performance of alternative x on criterion i is denoted x_i .

2.1. Overall '*at least as good as*' relation

In order to characterize between any two alternatives x and y of A a local '*at least as good as*' situation (Roy and Bouysoou, 1993; Bisdorff, 2002), with each criterion i is associated a double threshold order \geq_i whose bipolar characteristic representation $r(x \geq_i y)$ takes value:

$$\begin{array}{ll} +1 & \text{if } x_i + q_i \geq y_i ; \\ -1 & \text{if } x_i + p_i \leq y_i ; \\ 0 & \text{otherwise.} \end{array}$$

Furthermore, we associate with each criterion i in F a *rational significance* weight w_i which represents the contribution of i to the *overall warrant or not* of the *at least as good as* preference situation between all pairs of alternatives. Let W be the set of relative significance weights associated with F such that $W = \{w_i \mid i \text{ in } F\}$, with $0 < w_i < 1$ and $\sum_{i \text{ in } F} w_i = 1$.

Definition 2.1 The bipolar-valued characteristic representation r of the overall '*at least as good as*' relation (Bisdorff, 2000, 20002), denoted \geq , aggregating all the partial '*at least as good as*' situations \geq_i for i in F , is given by:

$$r(x \geq y) = \sum_{i \text{ in } F} [w_i \cdot r(x \geq_i y)] \quad (1)$$

For each criterion i in F , we can similarly characterize a local '*better than*' situation between any two alternatives x and y of A with a double threshold order $>_i$ and whose bipolar numerical representation $r(x >_i y)$ takes value:

$$\begin{array}{lll} +1 & \text{if} & x_i - p_i \geq y_i; \\ -1 & \text{if} & x_i - q_i \leq y_i; \\ 0 & & \text{otherwise.} \end{array}$$

Again, the overall '*better than*' relation is characterized by:

$$r(x > y) = \sum_{i \text{ in } F} [w_i \cdot r(x >_i y)] \quad (2)$$

Lemma 2.1 For each criterion i , the codual $(\neg \geq_i)^{-1}$ of the local '*at least as good as*' relation \geq_i on A is identical to the local '*better than*' relation $>_i$ on A .

Proof: Indeed, for all x, y in A , $r(\neg(x \geq_i y))$ equals:

$$\begin{array}{lll} -1, & \text{if} & y_i + q_i \geq x_i; \\ +1, & \text{if} & y_i + p_i \leq x_i; \\ 0, & & \text{otherwise.} \quad \square \end{array}$$

The lemma leads immediately to the following proposition.

Proposition 2.1

The overall '*better than*' relation $>$ on A is the codual, i.e. the converse of the negation, of the overall '*at least as good*' relation \geq on A .

Proof: Following lemma (2.1), formula (1) gives the same result for $(\neg \geq)^{-1}$ than formula (2) gives for $>$. \square

2.2 The classic veto principle

In order to characterize a local *veto* situation (Roy and Bouyssou, 1993) between any two alternatives x and y , we may associate with each criterion performance scale $[0; M_i]$ a *veto* (v_i) discrimination threshold such that $p_i < v_i \leq M_i + \varepsilon$ for all i in F (see [8]).

Definition 2.2 We may thus define on each criterion i a single threshold order denoted \ll_i which represents a '*seriously worse performing than*' situation (Bisdorff, 2008) and whose numerical representation $r(x \ll_i y)$ takes value:

$$\begin{array}{lll} +1, & \text{if} & x_i + v_i \leq y_i; \\ -1, & & \text{otherwise.} \end{array}$$

The characteristic representation of a (global) '*veto*' situation is now given by the overall disjunction of local '*seriously worse performing than*' situations:

$$r(x \ll y) := r(\bigvee_{i \text{ in } F} (x \ll_i y)) := \max_{i \text{ in } F} [r(x \ll_i y)]. \quad (3)$$

We are now ready to define the classic outranking relation.

2.3 The classic outranking relation

Definition 2.3 An alternative x '*outranks*' an alternative y , denoted $(x \succcurlyeq y)$, when:

1. a *significant majority* of criteria validates the fact that x is performing at least as good as y , i.e. $x \geq y$,
2. and there is no veto raised against this validation, i.e. $\neg(x \ll y)$.

The corresponding numerical representation gives:

$$r(x \succcurlyeq y) := r[(x \succcurlyeq y) \wedge \neg(x \llcurlyeq y)] := \min[r(x \succcurlyeq y), -r(x \llcurlyeq y)] \quad (4)$$

Proposition 2.3 (Pirlot & Bouyssou 2009)

Let \succcurlyeq be a classic outranking relation defined on a set of alternatives as shown in Definition 2.3 above.

1. The asymmetric part \succ of the classic outranking relation, i.e. $(x \succ y)$ and $\neg(y \succ x)$, is in general not identical to its codual relation $(\neg\succ)^{-1}$.
2. The absence of any veto situation is a sufficient and necessary condition for making \succ identical to $\neg(\preccurlyeq)$.

Proof:

(1) $r(\neg(y \succcurlyeq x)) = r[\neg((x \succcurlyeq y) \wedge \neg(x \llcurlyeq y))] = r[(\neg(x \succcurlyeq y) \vee \neg\neg(x \llcurlyeq y))] = \max[r(x \succ y), r(x \llcurlyeq y)]$ whereas $r(x \succcurlyeq y) = \min[r(x \succcurlyeq y), r(\neg(y \succcurlyeq x))]$. The strict inequality appears when $r(x \llcurlyeq y) = 1$.

(2) $v_i = M_i + \varepsilon$ implies that $r(x \succcurlyeq y) = r(x \succcurlyeq y)$ and the claimed identity follows from Proposition 2.1. Conversely, suppose that $v_i < M_i + \varepsilon$ and there exists a strong veto situation ($r(x \llcurlyeq_i y) = 1$) on some criterion i in F . In this case $\min[r(x \succcurlyeq y), r(\neg(y \succcurlyeq x))] = \min[-1, r(\neg(y \succcurlyeq x))] = -1 < r((y \succcurlyeq x)) = 1$. \square

As recently reported by Pirlot and Bouyssou (2009), this hiatus between the asymmetric definition and the codual relation raises a serious concern with respect to the logical soundness of the classic outranking definition. Only the absence of any veto mechanism can guarantee this somehow necessary property from the point of view of the intended semantics of the outranking concept. But this is vanishing the very interest of Roy's original outranking concept itself (Roy, 1991).

3. OUTRANKING WITH BIPOLAR VETO

3.1 The bipolar veto concept

From Proposition 2.3 we get the hint that the veto principle is in fact the concept that we have to put into a bipolar epistemic setting in order to overcome the previously mentioned hiatus.

Definition 3.1.a We may thus redefine on each criterion i a single threshold order denoted \lll_i which represents a '*seriously worse performing than*' situation and whose bipolar numerical representation $r(x \lll_i y)$ takes value:

$$\begin{aligned} &+1, \text{ if } x_i + v_i \leq y_i; \\ &-1, \text{ if } x_i - v_i \geq y_i; \\ &0, \text{ otherwise.} \end{aligned}$$

Similarly we may define on criterion i a single threshold order denoted \ggg_i representing a '*seriously better performing than*' situation and whose bipolar numerical representation $r(x \ggg_i y)$ takes value:

$$\begin{aligned} &+1, \text{ if } x_i - v_i \geq y_i; \\ &-1, \text{ if } x_i + v_i \leq y_i; \\ &0, \text{ otherwise.} \end{aligned}$$

It is worthwhile noticing that the bipolar negation is thus symmetrically opposing '*seriously better performing than*' to '*seriously worse performing than*' local veto situations and that each one of these relations is therefore the codual of the other. In case $v_i = M_i + \varepsilon$ again, the criterion i supports neither \lll_i nor \ggg_i situations.

Definition 3.1.b The bipolar characteristic representation of a (global) '*veto*' situation is now given by the aggregated determination of all local '*seriously worse performing than*' and '*seriously better performing than*' situations:

$$r(x \lll y) := \bigoplus_{i \in F} r(x \lll_i y). \quad (5)$$

$$r(x \ggg y) := \oplus_{i \in F} r(x \ggg_i y). \quad (6)$$

where \oplus represents the bipolar sharpening operator, called *epistemic disjunction* (see Grabisch et. al 2009, Bisdorff 1997) and defined as follows: $r \oplus r'$ equals $\max(r, r')$ if $r \geq 0$ and $r' \geq 0$; $\min(r, r')$ if $r \leq 0$ and $r' \leq 0$; and, 0 otherwise.

We may thus observe that $r(x \lll y) = 1$ iff there exists i in F such that $r(x \lll_i y) = 1$ and there does not exist any j in F such that $r(x \ggg_j y) = 1$. Or conversely, $r(x \ggg y) = 1$ iff there exists i in F such that $r(x \ggg_i y) = 1$ and there does not exist any j in F such that $r(x \lll_j y) = 1$.

Lemma 3.1

The bipolar codual $(\neg \lll)^{-1}$ of the global '*seriously worse performing than*' relation \lll on A is identical to the global '*seriously better performing than*' relation \ggg on A .

Proof: On each criterion i , the bipolar codual $(\neg \lll_i)^{-1}$ of the local '*seriously worse performing than*' relation \lll_i on A is identical to the local '*seriously better performing than*' relation \ggg_i on A . As the bipolar sharpening operator \oplus is auto-dual, it follows that the codual of the relation \lll is therefore the relation \ggg on A . \square

We may now define an outranking concept which is coherent with our bipolar approach.

3.2 The bipolar outranking relation

Definition 3.2. Let x and y be two decision alternatives. From a bipolar point of view, we say that :

1. x '*outranks*' y , denoted $x \succsim y$, if a significant majority of criteria validates a global outranking situation between x and y and no serious counter-performance is observed on a discordant criterion,
2. x '*does not outrank*' y , denoted $\neg(x \succsim y)$, if a significant majority of criteria invalidates a global outranking situation between x and y and no seriously better performing situation is observed on a concordant criterion.

In terms of our bipolar numeric representation r we obtain the following formal definition:

$$r(x \succsim y) := 0 \quad \text{if} \quad r(x \lll y) = 1 \text{ and } r(x \ggg y) = 1; \\ [r(x \geq y) \oplus -r(x \lll y)] \quad \text{otherwise.} \quad (5)$$

If $v_i = M_i + \varepsilon$ for all i in F , i.e. in the absence of any vetoes, we recover the previous case where $r(x \succsim y) = r(x \succ y) = r(x \geq y)$. If we observe conjointly seriously better and worse performances, the outranking statement gets indeterminate. If we observe a seriously better performing situation, $r(x \ggg y) = 1$, coupled with $r(x \geq y) \geq 0$, the outranking situation is certainly validated, that is we obtain $r(y \succ x) = 1$. Conversely, if we observe a seriously worse performing situation, $r(x \lll y) = 1$, coupled with $r(x \geq y) \leq 0$, the outranking situation is certainly not validated and we obtain $r(y \succ x) = -1$. Otherwise, we observe either $r(x \geq y)$ when $r(x \lll y) = 0$, or an indeterminate situation similar to the first case. The apparent preferential information, either $r(x \geq y) > 0$ with $r(x \lll y) = -1$, or $r(x \geq y) < 0$ with $r(x \lll y) = 1$, appears indeed contradictory and hence no positive or negative validation conclusion may be drawn from the epistemic aggregation.

Let us now show that the codual of the bipolar version of the outranking relation with the new extended veto principle is indeed identical with the corresponding strict bipolar outranking relation.

Let $(\neg \succsim)^{-1}$ denote the codual of the bipolar outranking relation, i.e. the converse of the bipolar negation of \succsim . If we define the strict bipolar outranking relation, denoted \succ , as follows:

$$r(x \succsim y) := 0 \quad \text{if} \quad r(x \lll y) = 1 \text{ and } r(x \ggg y) = 1;$$

$$[r(x > y) \oplus -r(x \lll y)] \quad \text{otherwise.} \quad (6)$$

we obtain the following result:

Proposition 3.2

$$r(\neg(x \succsim y)^{-1}) = r(x \succsim y) \quad \text{forall } (x, y) \text{ in } A^2 .$$

Proof:

$$\begin{aligned} r(\neg(x \succsim y)) &= -\{ [r(x \leq y) \oplus -r(x \ggg y)] \} \\ &= [-r(x \leq y) \oplus r(x \ggg y)] \quad (\text{The } \oplus \text{ operator is auto dual}) \\ &= [r(x > y) \oplus -r(x \lll y)] \quad (\text{Proposition 2.1, Lemma 3.1}). \quad \square \end{aligned}$$

3.3 Illustrative example

Let us consider performance evaluations (see Table 1) of five potential decision alternatives a_1, a_2, \dots, a_5 we may observe on a consistent family (see Roy & Bouyssou 1993) of five criteria.

Table 1. Random performance tableau

criteria	weight	a_1	a_2	a_3	a_4	a_5
g_1	0.04	76.32	40.35	71.22	76.99	28.10
g_2	0.30	24.97	40.00	24.99	16.07	29.67
g_3	0.33	19.00	70.68	36.63	87.98	8.90
g_4	0.22	86.83	1.28	14.71	91.46	55.43
g_5	0.11	29.88	33.43	65.62	15.69	57.01

All criteria: g_1, g_2, \dots, g_5 , with given significance weights (see Table 1), admit a real performance scale running from 0.0 (worst level) to 100.0 (best level) associated with three discrimination thresholds: *indifference* (± 10.0), *preference* ($+20.0$), and *seriously better or worse performing* (± 70.0). The resulting overall 'at least as good as' relation (see Definition 2.1.1) is shown below in Table 2, where we mention within brackets, the case given, the presence of a seriously better (+1), respectively worse (-1), performance.

Table 2. Overall 'at least as good as' relation with *seriously better or worse performing* denotation

$r(x \geq y)$ ($r(x \ggg y)$)	a_1	a_2	a_3	a_4	a_5
a_1	-	+0.04 (+1)	+0.44 (+1)	+0.33	+0.78
a_2	+0.48 (-1)	-	+0.48	+0.15 (-1)	+0.33
a_3	+0.56 (-1)	+0.04	-	-0.11 (-1)	+0.56
a_4	+0.89	+0.30 (+1)	+0.78 (+1)	-	+0.48 (+1)
a_5	+0.15	0.00	+0.30	-0.19 (-1)	-

The bipolar valuation may give figures between +1.00 (certainly 'at least as good as') and -1.00 (certainly not 'at least as good as'). For instance, $r(a_1 \geq a_2) = +0.04$ indicates that the preferential statement " a_1 is at least as good as a_2 " is only warranted by a short significance majority ($4.0 + 100.0 = 104.0 / 2 = 52\%$). However, we observe one seriously better performing situation on criterion g_4 , which helps confirm the warrant and we obtain $r(a_1 \succ a_2) = 1$. If we consider now the converse preferential statement " a_2 is at least as good as a_1 ", we may notice a much larger positive support ($r(a_2 \geq a_1) =$

+0.48). This high significance (nearly 75%) is however put to doubt by the seriously worse performance already observed on criterion g_4 and we obtain $r(a_2 \succsim a_1) = 0$.

Table 3. Bipolar outranking relation

$r(x \succsim y)$	a_1	a_2	a_3	a_4	a_5
a_1	-	+1.00	+1.00	+0.33	+0.78
a_2	0.00	-	+0.48	0.00	+0.33
a_3	0.00	+0.04	-	-1.00	+0.56
a_4	+0.89	+1.00	+1.00	-	+1.00
a_5	+0.15	0.00	+0.30	-1.00	-

The resulting complete valuation of the outranking relation is shown in Table 3. Five outranking situations: $(a_1 \succsim a_2)$, $(a_1 \succsim a_3)$, $(a_4 \succsim a_2)$, $(a_4 \succsim a_3)$, and $(a_4 \succsim a_5)$, get certainly validated. Whereas two outranking situations: $(a_3 \succsim a_4)$ and $(a_5 \succsim a_4)$, get certainly invalidated. Three outranking statements: $(a_2 \succsim a_1)$, $(a_2 \succsim a_4)$, as well as $(a_3 \succsim a_1)$, may neither be validated nor invalidated on the basis of the given performance tableau. Instead, the classical outranking relation would ignore the positive polarizations and mark the three last indeterminate statements as certainly false. Hence appears with classic outranking relations the hiatus between its asymmetric part and the corresponding codual relation.

4. CONCLUSION

In this paper we have introduced a new bipolar veto principle which allows us to construct an extended bipolar outranking relation guaranteeing the formal identity of the corresponding strict outranking relation, i.e. its asymmetric part, with its bipolar codual relation. Contrary to the classic unipolar outranking relation, taking into account only invalidating causes (via the classic veto principle) and where therefore incomparability situations potentially capture the difficulty to compensate seriously better performances with serious counter-performances, here we rely on the neutral value of the bipolar characteristic calculus for expressing our doubts concerning the effective compensation of such contrasted performances.

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