Inverse Analysis from a CONDORCET robustness denotation of valued outranking relation

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Outline

Motivation

In the outranking based MCDA (Roy 74), two different approaches exist to specify criteria significance weights:

0.1 either via *direct* knowledge or assessment
- Roy & Bouyssou 93;
- Roy & Mousseau 96,

0.2 or via some *a priori partial knowledge* of the resulting aggregated outranking is used:
- Mousseau & Słowinski 98;
- Meyer, Marichal & Bisdorff 08.

Here, we focus on the latter, the indirect preference information approach. Similar disaggregation-aggregation or ordinal regression methods have been proposed in MAUT and MAVT contexts:

- Jacquet-Lagrèze & Siskos 82;
- Mousseau, Figueira, Dias, Gomes da Silva & Clímaco 03;
- Greco, Mousseau & Słowinski 08;
- Grabisch, Kojadinovic & Meyer 08.

Our inverse analysis uses the robustness of the significant majority that the decision maker acknowledges for his/her pairwise outranking comparisons (Bisdorff 04).

Notations

- Let $A = \{x, y, z, \dots\}$ be a finite set of $n > 1$ potential decision alternatives
- and $F = \{g_1, \dots, g_m\}$ a coherent finite family of $m > 1$ real valued criteria functions.
- The performance of alternative x on criterion g_i is denoted x_i.
- To each g_i in F is associated an indifference q_i and a preference p_i discrimination threshold.
- This leads to a double threshold order S_i whose numerical representation is given by:

$$S_i(x, y) = \begin{cases} 1 & \text{if} \quad x_i + q_i \geqslant y_i \,, \\ -1 & \text{if} \quad x_i + p_i \leqslant y_i \,, \\ 0 & \text{otherwise.} \end{cases}$$

The bipolar-valued outranking relation

- $W = \{w_i : g_i \in F\}$ is a vector of normalized significance weights

  where w_i represents the contribution of g_i to the overall warrant or not of the at least as good as preference situation between all pairs of alternatives.
- The bipolar-valued outranking relation is defined as :

$$\widetilde{S}^W(x,y) = \sum_{w_i \in W} w_i \cdot S_i(x,y), \ \forall (x,y) \in A \times A.$$

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Bipolar semantics of the valued outranking

- $\widetilde{S}^W(x,y) = +1.0$ indicates that all criteria unanimously warrant the "*at least as good as*" preference situation;
- $\widetilde{S}^W(x,y) > 0.0$ indicates that a significant majority of the criteria warrant the "*at least as good as*" preference situation;
- $\widetilde{S}^W(x,y) = 0.0$ indicates a balanced situation;
- $\widetilde{S}^W(x,y) < 0.0$ indicates that a significant majority of criteria do not warrant the "*at least as good as*" preference situation;
- $\widetilde{S}^W(x,y) = -1.0$ indicates that all criteria unanimously warrant the negation of the "*at least as good as*" preference situation.

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The CONDORCET robustness denotation

- Let $\succsim_W$ be the preorder on F associated with the natural $\geqslant$ relation on the weights of the significance vector W.
- $\sim_W$ induces r ordered equivalence classes $\Pi_1^W \succ_W \ldots \succ_W \Pi_r^W$ $(1 \leq r \leq m)$.
- The criteria of an equivalence class have the same significance weight in W.
- For $i < j$, those of Π_i^W have a higher significance weight than those of Π_j^W.
- If $\mathcal{W}$ represents the set of all potential significance weights vectors, then $\mathcal{W}_{\succsim_W} \subset \mathcal{W}$ denotes the set of all significance weights vectors that are preorder-compatible with $\succsim_W$.

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The CONDORCET robustness denotation (continue)

The CONDORCET robustness $[\![\widetilde{S}^W]\!]$ of $\widetilde{S}^W$ is denoted as follows:

- $[\![\widetilde{S}^W]\!](x,y) = \pm 3$ if all criteria *unanimously warrant* (resp. *do not warrant*) the outranking situation between x and y;
- $[\![\widetilde{S}^W]\!](x,y) = \pm 2$ if a *significant majority of criteria warrants* (resp. *does not warrant*) the outranking situation between x and y for all $\succsim_W$-compatible weights vectors;
- $[\![\widetilde{S}^W]\!](x,y) = \pm 1$ if a significant majority of criteria *warrants* (respectively *does not warrant*) this outranking situation for W but not for all $\succsim_W$-compatible weights vectors;
- $[\![\widetilde{S}^W]\!](x,y) = 0$ if the total significance of the warranting criteria is *exactly balanced* by the total significance of the not warranting criteria for W.

Measuring the CONDORCET robustness

- Let $S_i^{\%} = (S_i + 1)/2$ be the $[0,1]$-recoded characteristic functions and let there be $k = 1, ..., r$ significance classes Π_k.
- Let $c_k^W(x,y)$ be the sum of "*at least as good as*" characteristics $S_i^{\%}(x,y)$ for all criteria $g_i \in \Pi_k^W$, and $\overline{c_k^W}(x,y)$ the sum of the negation $1 - S_i^{\%}(x,y)$ of these characteristics.
- Furthermore, let $C_k^W(x,y) = \sum_{i=1}^{k} c_i^W(x,y)$ be the cumulative sum of "*at least as good as*" characteristics for all criteria having significance at least equal to the one associated to Π_k^W, and

  let $\overline{C_k^W}(x,y) = \sum_{i=1}^{k} \overline{c_i^W}(x,y)$ be the cumulative sum of the negation of these characteristics for all k in $\{1,\ldots,r\}$.

Measuring the CONDORCET robustness (continue)

In the absence of ± 3 denotations, the following proposition gives us a test for the presence of a $+2$ denotation:

Proposition (Bisdorff 2004, *4OR:2(4)*)

$$[\![\widetilde{S}^W]\!](x,y) = +2 \iff \begin{cases} \forall k \in 1,...,r : C_k^W(x,y) \geqslant \overline{C_k^W}(x,y); \\ \exists k \in 1,...,r : C_k^W(x,y) > \overline{C_k^W}(x,y). \end{cases}$$

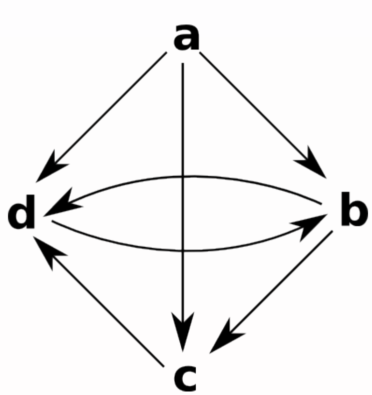
The negative -2 denotation corresponds to similar conditions with reversed inequalities.

The proof relies on the verification of first order stochastic dominance conditions.

Example of valued outranking

	g_1	g_2	g_3
a	10	4	8
b	5	6	4
c	7	2	3
d	5	7	2
p	1.0	1.0	1.0
W	3.0	1.5	2.0

$\widetilde{S}^W$	a	b	c	d
a	-	.54	1.0	.54
b	-.54	-	.08	.54
c	-1.0	-.08	-	.54
d	-0.54	0.38	-.54	-

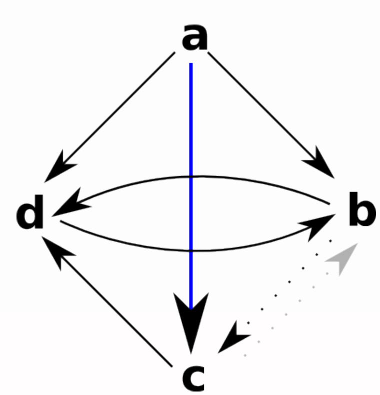


The CONDORCET Outranking Digraph

CONDORCET robustness

	g_1	g_2	g_3
p	1.0	1.0	1.0
W	**3.0**	**1.5**	**2.0**
a	10	4	8
b	5	6	4
c	7	2	3
d	5	7	2

$[\![\widetilde{S}^W]\!]$	a	b	c	d
a	-	2	**3**	2
b	-2	-	**1**	2
c	**-3**	**-1**	-	2
d	-2	2	-2	-

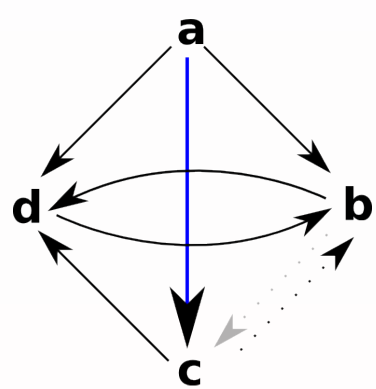


The CONDORCET Outranking Digraph

CONDORCET robustness

	g_1	g_2	g_3
p	1.0	1.0	1.0
W	**4.0**	**1.5**	**2.0**
a	10	4	8
b	5	6	4
c	7	2	3
d	5	7	2

$[\![\widetilde{S}^W]\!]$	a	b	c	d
a	-	2	3	2
b	-2	-	**-1**	2
c	-3	**1**	3	2
d	-2	2	-2	-



The CONDORCET Outranking Digraph

Inverse Analysis from the CONDORCET robustness

In a decision aid problem we are generally given

1. a performance table $A \times F$, but without any explicit significance weights information.
2. Suppose we are however given the apparent CONDORCET robustness denotation $[\![\widetilde{S}^W]\!]$, but with W and $\widetilde{S}^W$ actually unknown.

Inverse Analysis from the CONDORCET robustness

The inverse estimation problem

May we compute on the basis of the given information a preorder $\succsim$ on the criteria and a numerical instance W^* of a $\succsim$-compatible weights vector which satisfies the given CONDORCET robustness denotation $[\![\widetilde{S}^W]\!]$, i.e.

$$W^* \text{ and } \succsim \text{ are such that } \quad [\![\widetilde{S}^{W^*}]\!] = [\![\widetilde{S}^W]\!] \; ?$$

Estimating apparent criteria significance weights

The decision variables $P_{m\times M}$

- Every criterion gets an integer significance weight $w_i \in [1, M]$, where M denotes the maximal admissible value.
- $P_{m\times M}$ is a Boolean $(0,1)$-matrix, with general term $[p_{i,u}]$, that characterises row-wise the number of weight units allocated to criterion g_i such that: $\sum_{u=1}^{M} p_{i,u} = w_i$.
- As an example, if g_i has an integer weight of 3 and if we decide that $M = 5$, then the ith row of $P_{m\times 5}$ is given by $(1,1,1,0,0)$.
- Every weight w_i is strictly positive: $\sum_{g_i \in F} p_{i,1} = m$.
- The cumulative constraints require that:

$$p_{i,u} \geqslant p_{i,u+1} \quad (\forall i = 1, ..., m,\ \forall u = 1, ..., M-1).$$

The CONDORCET robustness constraint

The CONDORCET robustness test may be formulated as:

$$[\![\widetilde{\mathrm{S}}^{W}]\!](x,y) = 2 \iff \begin{cases} \forall u \in 1,\dots,\max w_i : C_u^{\prime W}(x,y) \geqslant \overline{C_u^{\prime W}}(x,y)\,; \\ \exists u \in 1,\dots,\max w_i : C_u^{\prime W}(x,y) > \overline{C_u^{\prime W}}(x,y)\,; \end{cases}$$

where $C_u^{\prime W}(x,y)$ (resp. $\overline{C_u^{\prime W}}(x,y)$) is the sum of all $S_i^{\%}(x,y)$ (resp. $\overline{S}_i^{\%} = 1 - S_i^{\%}(x,y)$) such that the significance weight $w_i \leq u$.

For all pairs $(x,y) \in A^2_{+2}$ we get

$$\sum_{g_i \in F} \Big(p_{i,u} \cdot \big[S_i^{\%}(x,y) - \overline{S}_i^{\%}(x,y) \big] \Big) \geqslant b_u(x,y),$$

where the $b_u(x,y)$ are Boolean $(0,1)$ variables for each pair of alternatives and each equi-significance level u in $\{1,\dots,M\}$,

which allow us to impose at least one case of strict inequality for each $(x,y) \in A^2_{\pm 2}$: $\sum_{u=1}^{m} b_u(x,y) \geqslant 1$.

The objective function

$$\min_{P_{m\times M}} O \;=$$

$$K_1\Big(\sum_{g_i\in F}\sum_{u=1}^{M} p_{i,u}\Big) \quad \text{Minimize the sum of the weights;}$$

$$-\; K_2\Big(\sum_{u=1}^{M}\Big(\sum_{(x,y)\in A^2_{\pm 2}} b_u(x,y)\Big)\Big) \quad \text{Maximise the } \pm 2 \text{ robustness;}$$

$$+\; K_3\Big(\sum_{(x,y)\in A^2_{\pm 1}} s^{\pm 1}(x,y)\Big) + K_4\Big(\sum_{(x,y)\in A^2_{0}} \big(s^0_+(x,y) + s^0_-(x,y)\big)\Big)$$

Comment

- *$s^{\pm 1}$ as well as $s^0_\pm$ are slack variables for softening, the case given, the ± 1 and 0 robustness constraints,*
- *$K_1 \dots K_4$ are parametric constants used for the correct hierarchical ordering of the four sub-goals.*

The mixed-integer MP model

MILP

Variables:

$p_{i,u} \in \{0,1\}$	$\forall g_i \in F,\ \forall u = 1,..,M$
$b_u(x,y) \in \{0,1\}$	$\forall (x,y) \in A^2_{\pm 2}, \forall u = 1,..,M$
$s^{\pm 1}(x,y) \geqslant 0$	$\forall (x,y) \in A^2_{\pm 1}$
$s^0_+(x,y) \geqslant 0\ ,\ s^0_-(x,y) \geqslant 0$	$\forall (x,y) \in A^2_0$

Parameters:

$K_i > 0$	$\forall i = 1 \dots 4$

Objective function:

$$\min \quad K_1\Big(\sum_{g_i\in F}\sum_{u=1}^{M} p_{i,j}\Big) - K_2\Big(\sum_{u=1}^{M}\sum_{(x,y)\in A^2_{\pm 2}} b_u(x,y)\Big) + K_3\Big(\textstyle\sum_{(x,y)\in A^2_{\pm 1}} s^{\pm 1}(x,y)\Big) + K_4\Big(\textstyle\sum_{(x,y)\in A^2_0}\big(s^0_+(x,y) + s^0_-(x,y)\big)\Big)$$

The mixed-integer MP model (continue)

Constraints:

$$\sum_{g_i\in F} p_{i,1} = m$$

$$p_{i,u} \geqslant p_{i,u+1} \qquad \forall g_i \in F,\ \forall u = 1,..,M-1$$

$$\sum_{g_i\in F}\Big(p_{i,u}\cdot\big[S_i^{\%}(x,y) - \overline{S}_i^{\%}(x,y)\big]\Big) \gtreqqless b_u(x,y) \qquad \forall (x,y) \in A^2_{\pm 2},\ \forall u = 1,..,M$$

$$\sum_{u=1}^{M} b_u(x,y) \geqslant 1 \qquad \forall (x,y) \in A^2_{\pm 2}$$

$$\sum_{g_i\in F}\Big(\big(\textstyle\sum_{u=1}^{M} p_{i,u}\big)\cdot \pm\big(S_i^{\%}(x,y) - \overline{S}_i^{\%}(x,y)\big)\Big) \pm s^1_\pm(x,y) \geqslant 1 \qquad \forall (x,y) \in A^2_{\pm 1},\ \forall u = 1,..,M$$

$$\sum_{g_i\in F}\big(\textstyle\sum_{u=1}^{M} p_{i,u}\big)\cdot\big(S_i^{\%}(x,y) - \overline{S}_i^{\%}(x,y)\big) + s^0_+(x,y) - s^0_-(x,y) = 0 \qquad \forall (x,y) \in A^2_0,\ \forall u = 1,..,M$$

Result of the Inverse Analysis

	g_1	g_2	g_3
p	1.0	1.0	1.0
W	3.0	1.5	2.0
a	10	4	8
b	5	6	4
c	7	2	3
d	5	7	2
W^*	**3.0**	**2.0**	**2.0**

Cond	a	b	c	d
a	-	2	3	2
b	-2	-	-1	2
c	-3	1	3	2
d	-2	2	-2	-

$\widetilde{S}^W$	a	b	c	d
a	-	.54	1.0	.54
b	-.54	-	.08	.54
c	-1.0	-.08	-	.54
d	-0.54	0.38	-.54	-

$\widetilde{S}^W$	a	b	c	d
a	-	.43	1.0	.43
b	-.43	-	.14	.43
c	-1.0	-.14	-	.43
d	-0.43	0.43	-.43	-

Valued outranking relation from estimated weight vector $[3, 2, 2]$.

Solving the MILP

- We solve the MILP model with Cplex 11.0, associated with an AMPL front end modeler;
- On more or less real-sized random multiple criteria decision problems (20 alternatives evaluated on 13 criteria) we observe quite reasonable solving times on an 6 threaded standard application server;
- Depending on the maximal value M allowed for an individual criterion significance weight we indeed obtain:
  - average computation times of 2.5 seconds for $M = 7$,
  - up to 2 minutes for $M = 13$.

Partial preference information

Partial preference information may be easily integrated in the previous MILP model, like

1. fix or confine the a priori significance of some criterion;
2. make a criterion, or a coalition of criteria, more significant than others;
3. allocate a significant majority to a coalition of criteria.

A progressive and robust decision aid approach

- When no information concerning the significance of the criteria is available, we solve the problem with equi-significant criteria, i.e. one single weight equivalence class.
- Some apparent outranking situations may be aknowledged, some others not. Under this partial preference information, the most robust valued outranking relation is estimated.
- As long as the resulting outranking digraph is too indeterminate, we may ask further partial preference information until the decision maker is satisfied with the overall result.

Concluding remarks

- We present an innovative approach for constructing criteria significance weights from the CONDORCET robustness of a bipolar-valued outranking relation.
- The corresponding MILP model may be solved in reasonable time for realistic decision aid problems.
- A new progressive and robust decision aid methodology may be based on an interactive and specificaly focused inverse MCDA.