On a bipolar foundation of the outranking concept

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Motivation

1. Let x and y be integers.
   - Either: $x < y$, or $x = y$, or $x > y$.
   - Thus, saying that $x \not\geqslant y$, means in fact that $y > x$.
   - Obviously, this is due to the fact that the ordering of integer numbers is *complete* !
2. Let x and y be two decision alternatives.
   - What does mean the sentence: *"x does not outrank y"* ?
   - Does it means that consequently *"y strictly outranks x"* ?
   - Not necessarily!
   - The classic outranking relation, due potential *veto* situations, may be *partial* only.

Notations

- $A = \{x, y, z, ...\}$ is a finite set of decision alternatives.
- $F = \{1, ..., n\}$ is a finite and coherent family of performance criteria.
- For each criterion i in F, the alternatives are evaluated on a real performance scale $[0; M_i]$,

  supporting an indifference threshold q_i

  and a preference threshold p_i such that $0 \leqslant q_i < p_i \leqslant M_i$.
- The performace of alternative x on criterion i is denoted x_i.

Performing *at least as good as* on a single criterion

Each criterion i is characterising a double threshold order $\geqslant_i$ on A in the following way:

$$r(x \geqslant_i y) = \begin{cases} +1 & \text{if} \quad x_i + q_i \geqslant y_i \\ -1 & \text{if} \quad x_i + p_i \leqslant y_i \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

$+1$ signifies x is *performing at least as good as* y on criterion i,

-1 signifies that x is *not performing at least as good as* y on criterion i.

0 signifies that it is *unclear* whether, on criterion i, x is performing at least as good as y.

Performing globally *at least as good as*

Each criterion i contributes the significance w_i of his "*at least as good as*" characterisation $r(\geqslant_i)$ to the global characterisation $r(\geqslant)$ in the following way:

$$r(x \geqslant y) = \sum_{i \in F} \left[w_i \cdot r(x \geqslant_i y) \right] \qquad (2)$$

- $r > 0$ signifies x is *globally performing at least as good as* y,
- $r < 0$ signifies that x is *not globally performing at least as good as* y,
- $r = 0$ signifies that it is *unclear* whether x is globally performing at least as good as y.

Performing *better than* on a single criterion

Each criterion i is characterising a double threshold order $>_i$ (*better than*) on A in the following way:

$$r(x >_i y) = \begin{cases} +1 & \text{if} \quad x_i - p_i \geqslant y_i \\ -1 & \text{if} \quad x_i - q_i \leqslant y_i \\ 0 & \text{otherwise.} \end{cases} \qquad (3)$$

And, the *global better than* relation is defined as:

$$r(x > y) = \sum_{i \in F} \left[w_i \cdot r(x >_i y) \right] \qquad (4)$$

First result

Proposition

The global better than relation ($>$) is the codual of the "global at least as good" ($\not\geqslant$) relation.

Proof.

On each criterion i:

$$r(x \not\geqslant_i y) = -r(x \geqslant_i y) = \begin{cases} -1 & \text{if} \quad x_i + q_i \geqslant y_i \\ +1 & \text{if} \quad x_i + p_i \leqslant y_i \\ 0 & \text{otherwise.} \end{cases} \quad . \qquad (5)$$

□

The classic veto principle

Roy introduced the concept of veto threshold v_i ($p_i < v_i \leqslant M_i + \epsilon$) to characterize the observation of seriously less performing situations on the family of criteria. This leads to a single threshold order, denoted $\ll_i$ which represents seriously less performing situations as follows:

$$r(x \ll_i y) = \begin{cases} +1 & \text{if} \quad x_i + v_i \leqslant y_i \\ -1 & \text{otherwise.} \end{cases} \quad . \qquad (6)$$

And a global veto situation $x \ll y$ is characterised as:

$$r(x \ll y) = r\Big(\bigvee_{i \in F} (x \ll_i y)\Big) = \max_{i \in F} \left[r(x \ll_i y) \right] \qquad (7)$$

The classic outranking relation

An alternative x *outranks* an alternative y , denoted $(x \succsim y)$, when:

1. a *significant majority* of criteria validates the fact that x is performing at least as good as s, i.e. $(x \geqslant y)$.
2. And, there is *no veto* raised against this claim, i.e. $\neg(x \ll y)$.

The corresponding charactistic gives:

$$
\begin{aligned}
r(x \succsim y) &= r\big[(x \geqslant y) \wedge \neg(x \ll y)\big] &(8)\\
&= \min\big[r(x \geqslant y), -r(x \ll y)\big] &(9)
\end{aligned}
$$

Second result

Proposition (Pirlot & Bouyssou 2009)

Let $\succsim$ be a classic outranking relation.

- *The asymmetric part $\succnsim$ of $\succsim$, i.e. $(x \succsim y)$ and $\neg(y \succsim x)$, is in general not identical to its codual relation $\not\precsim$.*
- *The absence of any veto situation is sufficient and necessary for making $\succnsim$ identical to $\not\precsim$.*

Seriously *better* or *worse performing* on a criterion

We redefine a single threshold order, denoted $\lll_i$ which represents *seriously less performing* situations as follows:

$$
r(x \lll_i y) = \begin{cases} +1 & \text{if} \quad x_i + v_i \leqslant y_i \\ -1 & \text{if} \quad x_i - v_i \geqslant y_i \\ 0 & \text{otherwise.} \end{cases} \qquad (10)
$$

And a corresponding dual *seriously better performing* situation $\ggg_i$ characterised as:

$$
r(x \ggg_i y) = \begin{cases} +1 & \text{if} \quad x_i - v_i \geqslant y_i \\ -1 & \text{if} \quad x_i + v_i \leqslant y_i \\ 0 & \text{otherwise.} \end{cases} \qquad (11)
$$

Gloablly seriously *better* or *worse performing*

A global veto, or counter-veto situation is now defined as follows:

$$
\begin{aligned}
r(x \lll y) &= \circledvee_{i \in F}\, r(x \lll_i y) &(12)\\
r(x \ggg y) &= \circledvee_{i \in F}\, r(x \ggg_i y) &(13)
\end{aligned}
$$

where $\circledvee$ represents the epistemic polarising (Bisdorff 1997) aggregation operator (Grabisch et al. 2009):

$$
r \circledvee r' = \begin{cases} \max(r, r') & \text{if} \quad r \geqslant 0 \wedge r' \geqslant 0, \\ \min(r, r') & \text{if} \quad r \leqslant 0 \wedge r' \leqslant 0, \\ 0 & \text{otherwise.} \end{cases} \qquad (14)
$$

Characterising very large performance differences

1. $r(x \lll y) = 1$ iff there exists a criterion i such that $r(x \lll_i y) = 1$ and there does not exist otherwise any criteria j such that $r(x \ggg_j y) = 1$.
2. Conversely, $r(x \ggg y) = 1$ iff there exists a criterion i such that $r(x \ggg_i y) = 1$ and there does not exist otherwise any criteria j such that $r(x \lll_j y) = 1$.
3. $r(x \ggg y) = 0$ if either we observe no very large perforemance differences or we observe at the same tiem, both a very large positive and a very large negative performance difference.

Lemma

$r(\not\lll)^{-1}$ is identical to $r(\ggg)$.

The bipolar outranking concept

From an *epistemic point of view*, we say that:

1. *x outranks y*, denoted $(x \succsim y)$, if a significant majority of criteria validates a global outranking situation between x and y and no serious counter-performance is observed on a discordant criterion,
2. *x does not outrank y*, denoted $(x \not\succsim y)$, if a significant majority of criterxia invalidates a global outranking situation between x and y and no seriously better performing situation is observed on a concordant criterion.

Polarising the global "*at least as good as*" characteristic

The bipolar characteristic $r(\succsim)$ is defined as follows:

$$r(x \succsim y) = \begin{cases} 0 & \text{if } \big[\exists i \in F : r(x \lll_i y)\big] \wedge \big[\exists j \in F : r(x \ggg_j y)\big] \\ \big[r(x \geqslant y) \circledvee -r(x \lll y)\big] & \text{otherwise} \end{cases}$$

And in particular,

- $r(x \succsim y) = r(x \geqslant y)$ if no very large positive or negative performance differences are observed,
- $r(x \succsim y) = 1$ if $r(x \geqslant y) \geqslant 0$ and $r(x \ggg y) = 1$,
- $r(x \succsim y) = -1$ if $r(x \geqslant y) \leqslant 0$ and $r(x \lll y) = 1$,

Final result

Proposition

The codual $(\not\succsim)^{-1}$ of the bipolar outranking relation $\succsim$ is identical to the strict outranking $\succnsim$ relation.

Proof.

$$\begin{aligned} r(x \not\succsim y) &= -r(x \succsim y) = -\big[r(x \geqslant y) \circledvee -r(x \lll y)\big] \\ &= \big[-r(x \geqslant y) \circledvee r(x \lll y)\big] \\ &= \big[r(x \not\geqslant y) \circledvee -r(x \ggg y)\big] \\ &= \big[r(y > x) \circledvee -r(y \lll y)\big] = r(y \succnsim x). \end{aligned}$$

□

Concluding ...

- We have shown that the strict version of the classic outranking is not identical with its codual.
- This is due to the unipolar definition of the veto principle.
- When considering an extended bipolar veto and counter-veto principle one gets back this identity.
- Time for a didactical example ... ?.