On Computing kernels on $\mathcal{L}$-valued simple graphs

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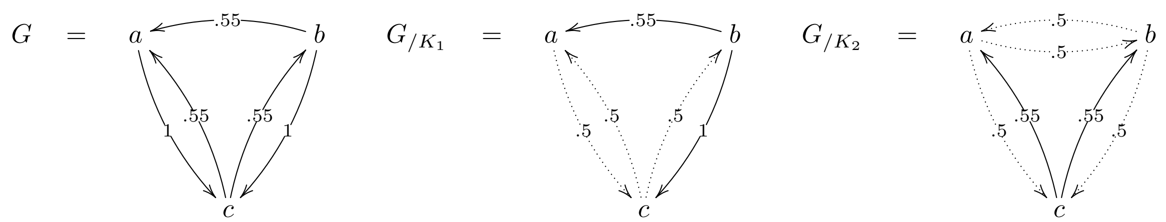
September 1997

ABSRACT: In this extended abstract we will present a decomposition of a given $\mathcal{L}$-valued binary relation into a set of $\mathcal{L}$-sub-relations of kernel-dimension one. We will apply this theoretical result to the design of a fast algorithm for computing $\mathcal{L}$-valued kernels on general $\mathcal{L}$-valued simple graphs.

1 INTRODUCTORY EXAMPLE

Recent research work (cf. Bisdorff & Roubens 1996 [1] [2]) on defining and computing fuzzy kernels from $\mathcal{L}$-valued simple graphs has shown that multiple kernel solutions correspond in fact to multiple possibilities for extracting independent alignments, that is fuzzy a-cyclic binary relations from given general fuzzy simple graphs.

To give a first intuitive illustration of this idea consider the following numerical example[1]:



This fuzzy out-ranking graph (Roy & Bouyssou [6]) G admits two fuzzy dominating kernel (cf. Bisdorff & Roubens [1], [2]) solutions $K_1 = \{a(.45), b(1), c(0)\}$, $K_2 = \{a(.45), b(.45), c(.55)\}$, to which correspond two “independent” restrictions, $G_{/K_1}$ and $G_{/K_2}$, of the original relation with $K_1 = \{a(.45), b(1), c(0)\}$ and $K_2 = \{a(.45), b(.45), c(.55)\}$ as respective unique fuzzy kernel solutions.

This example suggests that multiple kernel solutions are in fact hints towards a possible “boolean additive” decomposition of the original $\mathcal{L}$-valued relation.

2 ON “BOOLEAN ADDITION” OF $\mathcal{L}$-VALUED BINARY RELATIONS

Let $G^{\mathcal{L}} = (A, R)$ be a simple finite $\mathcal{L}$-valued graph, where A is a finite set and R an $\mathcal{L}$-valued relation on A. $\mathcal{L} = (V, \leqslant, \max, \min, \neg, \rightarrow, 0, 1)$ is a symmetric rational evaluation domain (cf. [1]), where the set V is a completely ordered possibly infinite set of rational truth values with ’0’ as “certainly false” and ‘1’ as “certainly true” value, ’$\neg$’ is a strong negation and ’$\rightarrow$’ is a residual min-implication[2]. We shall denote in the sequel $\mathcal{B} = (\{0, 1\}, \leqslant, \max, \min, \neg, \rightarrow, 0, 1)$ the standard boolean crisp logical denotation and $\mathcal{L}_3 = (\{0, \frac{1}{2}, 1\}, \leqslant, \max, \min,$-$\neg, \rightarrow, 0, 1)$ the ‘minimum’ trivalent symmetric evaluation domain (Bisdorff & Roubens [1], Bolc & Borowik [3], Fodor & Roubens [4]).

[1]This example was proposed by Bernard Roy at the 41th meeting of the EURO working group MCAD on “Multi-criteria Aid for Decisions” in Lausanne, March 1995 on the occasion of a lecture by Marc Roubens on the work about fuzzy kernels from Leonid Kitaïnik [5].

[2]We will follow in this research report the terminology introduced in Bisdorff, Roubens 1996a and 1996b.

Let $\mathcal{R}^{\mathcal{L}}_{AB}$ be the set of all possible $\mathcal{L}$-valued relations defined between given finite sets A and B. We shall denote M the all median- or $\frac{1}{2}$-valued relation. We may define on $\mathcal{R}^{\mathcal{L}}_{AB}$ a pair of dual "additive" operators '$\oplus$' and '$\ominus$' in the following manner:

Definition 1. Let $R, S \in \mathcal{R}^{\mathcal{L}}_{AB} : \forall (a,b) \in A \times B :$

$$(R \oplus | \ominus S)(a,b) = \begin{cases} \max | \min(R(a,b), S(a,b) & \Leftrightarrow \quad R(a,b) \geqq \frac{1}{2} \wedge S(a,b) \geqq \frac{1}{2}, \\ \min | \max(R(a,b), S(a,b) & \Leftrightarrow \quad R(a,b) \leqq \frac{1}{2}, \wedge S(a,b) \leqq \frac{1}{2}, \\ \frac{1}{2} \quad \text{elsewhere.} \end{cases}$$

In order to introduce the algebraic structures these operators define on $\mathcal{R}^{\mathcal{L}}_{AB}$, we need to introduce the median β-cut defuzzification.

Definition 2. Let $R \in \mathcal{R}^{\mathcal{L}}_{AB}$ be an $\mathcal{L}$-vbr defined between finite sets A and B. We note $R_{>\beta}$ the $\mathcal{L}_3$-valued β-cut relation constructed from R as follows: $\forall (a,b) \in A \times B$ and $\beta \in [\frac{1}{2}, 1[$,

$$R_{>\beta}(a,b) = \begin{cases} 1 & \Leftrightarrow \quad R(a,b) > \beta, \\ 0 & \Leftrightarrow \quad R(a,b) < \neg\beta, \\ \frac{1}{2} & \Leftrightarrow \quad \neg\beta \leqq R(a,b) \leqq \beta \end{cases}$$

In the sequel, we are mainly concerned by the median β-cut relation, denoted $R_{>\frac{1}{2}}$.

Proposition 1. *If $M \in \mathcal{R}^{\mathcal{L}}_{AB}$ is the trivial all median-valued relation, the algebraic structure $(\mathcal{R}^{\mathcal{L}}_{AB}, \oplus, M, R_{>\frac{1}{2}})$ gives an Abelian group with relation M as null element and relation $R_{>\frac{1}{2}}$ as absorbent element, whereas $(\mathcal{R}^{\mathcal{L}}_{AB}, \ominus, R_{>\frac{1}{2}}, M)$ gives an Abelian monoïd with the $R_{>\frac{1}{2}}, M)$ as null element and the trivial M as absorbent element. Furthermore, the $\oplus$- and $\ominus$-operators are mutually distributive.*

Proof. Indeed, the trivial fuzziest $\mathcal{L}$-valued relation M is a neutral element for the $\oplus$-operator and the maximal $\mathcal{L}_3$-valued $R_{>\frac{1}{2}}$ gives a neutral element for the $\ominus$-operator. Commutativity and double distributivity come naturally from the standard conjunction and disjunction operators min and max. Finally we may associate to every given relation $R \in \mathcal{R}^{\mathcal{L}}_{AB}$ its "contradictory" relation $\neg R = (1 - R) \in \mathcal{R}^{\mathcal{L}}_{AB}$ so that $R \oplus \neg R = M$. □

These operators are linked to the sharpness ordering (cf. Bisdorff & Roubens [1]) on $\mathcal{R}^{\mathcal{L}}_{AB}$ in the following way:

Definition 3. Let $R : A \times B \to V$ and $S : A \times B \to V$ be two $\mathcal{L}$-vbr's. We say that R is *sharper* than S, noted $R \succcurlyeq S$ iff $\forall (a,b) \in A \times B$: either $(R(a,b) \leqq S(a,b) \leqq \frac{1}{2})$ or $\frac{1}{2} \leqq S(a,b) \leqq R(a,b))$.

The sharpness relation '$\succcurlyeq$' on the set $\mathcal{R}^{\mathcal{L}}_{AB}$ of all $\mathcal{L}$-vbr's defined between any finite sets A and B of respective dimensions n and p, gives a partial order $(\mathcal{R}^{\mathcal{L}}_{AB}, \succcurlyeq)$ with the constant median-valued relation M as unique minimum element and $R^{\mathcal{B}}$, the 2^{np} possible $\mathcal{B}$-valued crisp relations between sets A and B, as the set of maximal (sharpest) elements.

Proposition 2. *Let $R, S \in \mathcal{R}^{\mathcal{L}}_{AB}$ be two $\mathcal{L}$-vbr's defined between sets A and B.*

$$R \succcurlyeq S \quad \Rightarrow \quad (R \oplus S \succcurlyeq R) \wedge (R \oplus S \succcurlyeq S), \tag{2.1}$$
$$R \succcurlyeq S \quad \Rightarrow \quad (R \succcurlyeq R \ominus S) \wedge (S \succcurlyeq R \ominus S). \tag{2.2}$$

Proof. Indeed, the boolean addition or subtraction of two $\succcurlyeq$-comparable relations gives an overall sharper respective fuzzier relation as result. This is an immediate consequence of definition 3 and definition 1. □

But we may more precisely isolate the subset of $\mathcal{L}$-valued relations which will give a $\oplus$-$\ominus$-lattice.

Definition 4. Let $R, S \in \mathcal{R}^{\mathcal{L}}_{AB}$ be two $\mathcal{L}$-vbr's defined between sets A and B. We say that R and S are **non-contradictory** iff $\forall (a,b) \in A \times B : \big(R(a,b) \leqq \frac{1}{2} \Rightarrow S(a,b) \leqq \frac{1}{2}\big) \wedge \big(\frac{1}{2} \leqq R(a,b) \Rightarrow \frac{1}{2} \leqq S(a,b)\big)$. We shall note this relation as $R \cong S$.

Or "non-contradiction" is precisely the restriction functor on $\mathcal{L}$-valued relations we look for.

Proposition 3. *Let $R, S \in \mathcal{R}^{\mathcal{L}}_{AB}$ be two $\mathcal{L}$-vbr's defined between finite sets A and B.*

$$R \cong S \quad \Leftrightarrow \quad (R \oplus S \succcurlyeq R) \wedge (R \oplus S \succcurlyeq S). \tag{3.1}$$
$$R \cong S \quad \Leftrightarrow \quad (R \succcurlyeq R \ominus S) \wedge (S \succcurlyeq R \ominus S). \tag{3.2}$$

Proof. Verification of these properties is straight forward. □

But then, we may notice that these additive operators $\oplus$ and $\ominus$ define in fact on a given $\cong$-congruence class of $\mathcal{R}^{\mathcal{L}}_{AB}$, a lattice structure with its median β-cut relation as top element and the all median-valued relation M as bottom element.

Proposition 4. *Let $R \in \mathcal{R}^{\mathcal{L}}_{AB}$ be an $\mathcal{L}$-vbr's defined between finite set A and B. Let $\mathcal{R}^{\mathcal{L}}_{/\cong R}$ be the "non-contradictory" congruence class associated to this relation R. If $M \in \mathcal{R}^{\mathcal{L}}_{A}$ is the trivial all median-valued relation, and $R_{>\frac{1}{2}}$ its associated median β-cut relation, the algebraic structure $\left(\mathcal{R}^{\mathcal{L}}_{A}, \oplus, \ominus, M, R_{>\frac{1}{2}}\right)$ gives a distributive lattice with relation M as bottom and $R_{>\frac{1}{2}}$ as top element.*

Proof. The $\oplus$-operator acts as lattice-join and the $\ominus$-operator as lattice-meet in a product of symmetric median-folded $\mathcal{L}$-algebras restricted to non-contradictory relations. The $\ominus$-operator has a lower limit which is the trivial all-median-valued relation M, whereas the $\oplus$-operator has an upper limit respective in the median β-cut relation. Double distributivity of $\oplus$ and $\ominus$ easily follows from the distributivity of the max and min operators of the underlying truth calculus algebra[3]. □

This results will allow us to state the main theorem of this report.

3 "ADDITIVE" DECOMPOSITION OF $\mathcal{L}$-VALUED RELATIONS

Let R be an $\mathcal{L}$-vbr defined on a given finite set A and let $K(R) = \{K_1, K_2, \ldots, K_n\}$ be its n fuzzy $\mathcal{L}$-valued kernel solutions. To each kernel solution $K_i, (i = 1, \ldots n)$ corresponds a specific $\mathcal{L}$-subgraph $G_{/K_i} = (A, R_{/K_i})$ defined in the following way.

Definition 5. Let K_i be a specific kernel solution on a given graph $G^{\mathcal{L}} = (A, R)$. We call the relation $R^d_{/K_i}$ defined as follows: $\forall (a, b) \in A \times A$:

$$R_{/K_i}(a,b) = \begin{cases} R(a,b) & \Leftrightarrow \begin{cases} \big((K_i(a) > \frac{1}{2}) \wedge (K_i(b) > \frac{1}{2}) \wedge (R(a,b) < \frac{1}{2})\big) & \vee \\ \big((K_i(a) > \frac{1}{2}) \wedge (R(a,b) > \frac{1}{2})\big) & \vee \\ \big((K_i(b) > \frac{1}{2}) \wedge (R(a,b) < \frac{1}{2})\big) & \vee \\ (a = b) \end{cases} \\ \frac{1}{2} & \text{elsewhere.} \end{cases}$$

a (dominant) kernel- or K_i-restriction of the original relation R (the respective absorbent kernel restriction is obtained similarly by inverting a and b in the formula above). We shall denote $R_{/K(R)} = \{R_{/K_i} \mid K_i \in K(R)\}$ the set of kernel restrictions on R based on the set $K(R)$ of possible kernel solutions obtained from R.

The first row integrates in the restricted relation all arrows describing the interior stability of the given kernel solution K_i. The second and third row integrate all arrows supporting the dominant (respectively the absorbent) exterior stability of the kernel solution in question. The diagonal terms are left unchanged and finally, all remaining arrows are put to the trivial undetermined truth value $\frac{1}{2}$.

Let us now investigate the relation-ship between these kernel-restrictions and the original relation R.

Proposition 5. *Let R be an $\mathcal{L}$-vbr defined on a given finite set A and let $K(R) = \{K_1, K_2, \ldots, K_n\}$ be its n fuzzy $\mathcal{L}$-valued kernel solutions.*

(i) *To each specific kernel solution K_i corresponds a unique K_i-restriction $R_{/K_i}$.*

(ii) *Every K_i-restriction $R_{/K_i}$ gives an $\mathcal{L}$-alignment.*

[3]The $\oplus, \ominus$-structure is not directly a boolean algebra (a complemented distributive lattice), as the complement of a given relation is in general not unique. We denote these structures as "projectively" boolean.

(iii) *All K_i-restrictions belong to the same $\cong$-congruence class $\mathcal{R}^{\mathcal{L}}_{/\cong R}$ determined by the original relation R and*

$$R_{/K_i} = \inf\{R' \in \mathcal{R}^{\mathcal{L}}_{\cong R} | K(R \ominus R') = K_i\}$$

(iv) *Let K_i and K_j be two kernel solutions from a given $\mathcal{L}$-valued relation R without any common kernel member. The K_i-restriction is then "independent" from the K_j-restriction in the following sense:*

$$R_{/K_i} \ominus R_{/K_j} = M \Leftrightarrow \forall a \in A : K_i(a) > \frac{1}{2} \Rightarrow K_j(a) \leqq \frac{1}{2}$$

Proof. (i) Consider $R_{/K_i}$ and $R'_{/K_i}$ being two different kernel restrictions of R corresponding to a same given kernel solution K_i. Then there $\exists (a,b) \in A \times A : R_{/K_i}(a,b) \neq R'_{/K_i}(a,b)$. This contradicts the construction principle of definition 5. (ii) From definition 5, it follows that the only strict $\mathcal{L}$-true elements of $R_{/K_i}$ are those coming from kernel members. All other elements are either $\mathcal{L}$-untrue or $\mathcal{L}$-undetermined. (iii) The restriction construction principle in definition 5 does give only non-contradictory sub-relations of R, in fact the smallest one in the sense of $\ominus$, such that this restriction does indeed support the corresponding kernel K_i. Indeed, only the specific arrows describing the interior and exterior stability of this specific kernel solution K_i are integrated in the restriction, all others are put to the trivial undetermined value $\frac{1}{2}$. (iv) This property is a direct consequence of the preceeding property in the sense that any two kernel solutions without any common kernel member will have no common $\mathcal{L}$-true elements. □

We may now formulate the principal theorem concerning the additive decomposition of a give $\mathcal{L}$-valued relation.

Theorem 1. *Let $G^{\mathcal{L}} = (A, R)$ be a finite $\mathcal{L}$-valued simple graph and $K(R) = \{K_1, K_2, \ldots, K_n\}$ the corresponding kernel solutions set. Let $R_{/K(R)}$ be the complete set of kernel restrictions corresponding to the set $K(R)$ of kernel solutions on R. The following relations are then verified:*

$$\hat{R} = \left[\bigoplus_{i=1}^{n} R_{/K_i}\right] \preccurlyeq R \quad (1.1)$$

$$K(R_{/K_i}) = \{K_i\}, i = 1, \ldots n \quad (1.2)$$

$$K(\hat{R}) = \bigcup_{i=1}^{n} K(R_{/K_i}) = K(R). \quad (1.3)$$

Proof. Every relation R may be decomposed into n relations $R_{/K_i}$ so that the $\oplus$-sum equals again a relation of same shape than the original relation R as a consequence of proposition 5.iii, and the kernel solution set for each relation $R_{/K_i}$ is exactly equal to the corresponding unique kernel solution K_i determined on the original relation R. This follows indeed from the K_i-restriction construction principle of definition 5, from proposition 5.i and from the partial monotonicity of the kernel construction w.r.t. the sharpness ordering "$\succcurlyeq$" on $\mathcal{R}^{\mathcal{L}}_{\cong R}$. Finally, the set-union of all individual kernel solutions restores back again the complete set $K(R)$ of kernel solutions observed on R. □

Example 1. Let $G^{\mathcal{L}} = (A, R)$ with R being an $\mathcal{L}$-empty ($\leqslant \frac{1}{2}$) relation, that is every credibility value observed in R is either $\mathcal{L}$-untrue or $\mathcal{L}$-undetermined (cf. Bisdorff & Roubens [1]). Such a relation has a unique fuzzy kernel solution K with every element undetermined or $\mathcal{L}$-true selected. As the kernel-restriction is neutral for such a relation, properties of theorem 1 are trivially verified in this case.

Example 2. On the other hand let us take the following $\mathcal{L}$-clique of dimension 3 (cf. Bisdorff & Roubens [1]), that is a relation on $A = \{a, b, c\}$ with every term being $\mathcal{L}$-true.

$$R = \begin{bmatrix} 1 & .75 & .75 \\ .70 & 1 & .70 \\ .60 & .60 & 1 \end{bmatrix}, \; R_{/K_1} = \begin{bmatrix} 1 & \mathbf{.75} & \mathbf{.75} \\ .50 & 1 & .50 \\ .50 & .50 & 1 \end{bmatrix}, \; R_{/K_2} = \begin{bmatrix} 1 & .50 & .50 \\ \mathbf{.70} & 1 & \mathbf{.70} \\ .50 & .50 & 1 \end{bmatrix}, \; R_{/K_3} = \begin{bmatrix} 1 & .50 & .50 \\ .50 & 1 & .50 \\ \mathbf{.60} & \mathbf{.60} & 1 \end{bmatrix}.$$

Such a relation has 3 fuzzy kernel solutions, $K(R) = \{K_1, K_2, K_3\}$ with $K_1 = \{\mathbf{a}(\mathbf{.75}), b(.25), c(.25)\}, K_2 = \{a(.30), \mathbf{b}(\mathbf{.70}), c(.30)\}$ and $K_3 = \{a(.40), b(.40), \mathbf{c}(\mathbf{.60})\}$, describing each a different fuzzy kernel singleton. The corresponding K_i-restrictions give $\mathcal{L}$-valued relations with each time a different row of credibility values different from $\frac{1}{2}$. To each such K_i-restriction corresponds a unique individual kernel solution. In this example, the additive recomposition $\hat{R} = R_{/K_1} \oplus R_{/K_2} \oplus R_{/K_3}$ is in fact identical to R so that properties 1.1 and 1.3 of theorem 1 are again trivially true.

Example 3. A third example is naturally given by Roy's out-ranking relation as shown in the introductory section. It is worthwhile noticing that in this case the recomposition does not restitute the original relation in its integrality. Indeed,

$$R = \begin{bmatrix} 1 & 0 & \mathbf{1} \\ .55 & 1 & 1 \\ .55 & .55 & 1 \end{bmatrix}, \qquad \hat{R} = R_{/K_1} \oplus R_{/K_2} = \begin{bmatrix} 1 & 0 & \mathbf{.50} \\ .55 & 1 & 1 \\ .55 & .55 & 1 \end{bmatrix},$$

and the difference appears in the value of the arrow from alternative $\{a\}$ to alternative $\{c\}$, where the original certain truth value '1' is not restored. But the re-composed relation is the smallest relation in the sense of $\ominus$ which is globally sharper in the sense of $\succcurlyeq$ to all K_i-restrictions and therefore supports again the complete set of kernel solutions on the original relation R.

This last example suggests the following theorem:

Theorem 2. *Let $G^{\mathcal{L}} = (A, R)$ be a finite $\mathcal{L}$-valued simple graph and $K(R) = \{K_1, K_2, \ldots, K_n\}$ the corresponding kernel solutions set. Let $R_{/K(R)}$ be the complete set of kernel restrictions corresponding to the set $K(R)$ of kernel solutions on R. Then*

$$R_{/K(R)} = \inf\{R' \in \mathcal{R}^{\mathcal{L}}_{\cong R} \mid K(R \ominus R') = K(R)\}$$

Proof. Indeed, each K_i-restriction only keeps those arrows different from the trivial undetermined value, that are necessary to describe the stability conditions of the given kernel solution K_i (cf. definition 5). The $\oplus$-sum of these K_i-restriction gives then as a consequence of proposition 4 the l.u.b. of these relations in the given $\mathcal{R}^{\mathcal{L}}_{\cong R}$ congruence class. □

General characterisation of kernel solutions sets for given $\mathcal{L}$-valued relations on a finite set A, has shown (cf. Bisdorff & Roubens [1]) that the kernel solution space is organised as lower-closed chains in the $\succcurlyeq$-sense from the maximal kernel solutions to the trivial median-valued undetermined solution. Or our "additive" decomposition allows us now to split this kernel solution space into independent relational 'fibres' associated to each individual kernel solution and collate these 'fibres' back again to cover sufficiently the original solution space so as to be sure to recover a relation supporting the complete set of original kernel solutions.

The above noted theoretical results suggest the design of a faster algorithm for $\mathcal{L}$-valued kernel computations as originally proposed by constrained finite domains enumeration (cf. Bisdorff & Roubens [2]).

4 A FAST ALGORITHM FOR COMPUTING $\mathcal{L}$-VALUED KERNELS

Indeed, the additive decomposition of a given relation R into a set of mono-nuclear fuzzier sub-relations allows to use for each such computation the very fast dual fix-point algorithm as originally proposed by Von Neumann (1944)[9] (see also Schmidt & Ströhlein [8], [7]), and applied to the fuzzy case by Kitaïnik ([5]).

First, we may observe that the kernel shapes given by the kernel solution from median β-cut relation are sufficient to determine the K_i-restrictions for a given general $\mathcal{L}$-valued relation R.

Proposition 6. *Let R be an $\mathcal{L}$-valued relation and let $K(R)$ be its associated kernel solutions set of dimension n. Let $R_{>\frac{1}{2}}$ be the associated β-cut relation. Let $K(R_{>\frac{1}{2}})$ be the corresponding $\mathcal{L}_3$-valued kernel solutions sets. Then $R_{/K(R)} = R_{/K(R_{>\frac{1}{2}})}$.*

Proof. The median β-cut is a natural transformation (in the categorical sense) from general $\mathcal{L}$-valued relation to $\mathcal{L}_3$-valued relations (cf. Bisdorff & Roubens [1]), so that the median β-cut kernel solutions are $\succcurlyeq$-comparable limits. Or definition 5 of the K_i-restriction does not rely on a precise value, but only on the $\succcurlyeq$-comparable shape of the kernel solution, so that K_i-restrictions of R from $K(R)$ and from $K(R_{>\frac{1}{2}})$ are in fact identical. □

Furthermore, from theorem 1 we know that any K_i-restriction has a unique kernel solution as it is an $\mathcal{L}$-alignment (Bisdorff & Roubens [1]). Or, application of the dual fix-point algorithm to such $\mathcal{L}$-alignments $R_{/K_i}$ for calculating maximal kernel solutions gives following result:

Proposition 7. *Let R be a given $\mathcal{L}$-alignment defined on a finite set A of dimension m and $K(R)$ its $\mathcal{L}$-valued kernel solution. Let Y be a possible $\mathcal{L}$-valued kernel-membership function and $R^{-1} \circ Y = \overline{Y}$ and $\overline{R^{-1} \circ Y'} = Y'$ be the dual anti-eigenvalue equations expressing the stability conditions of the dominant kernel construction*[4]*. Let the initial solution for Y be the all 0-valued vector and for Y' be the all 1-valued vector. The dual iteration will exchange at each step the corresponding values of Y and Y' in the two equations in order to reach rapidly in a finite number of steps (bound by m) two fix-points $\hat{Y}$ and $\hat{Y}'$ such that $K(R) = \hat{Y} \oplus \hat{Y}'$.*

Proof. Demonstration of this rather technical property of the dual fix-point algorithm is given by Kitaïnik ([5]). □

The general algorithm we propose is the following:

Definition 6 (Fast algorithm for computing $\mathcal{L}$-valued kernel solutions)**.**

$K(R) \leftarrow$ **kernelsolutions**($\mathcal{L}$-vbr R)
step 1:
  $K(R_{>\frac{1}{2}}) \leftarrow$ median β-cut kernel solutions on R
step 2:
  $R^K \leftarrow$ set of K_i-restrictions from $K(R_{>\frac{1}{2}})$
step 3:
  for $R_{/K_i} \in R^K$
    $K_i \leftarrow$ dual fix-point kernel computation on $R_{/K_i}$
  endfor
output $\cup_i K_i$
endkernelsolutions

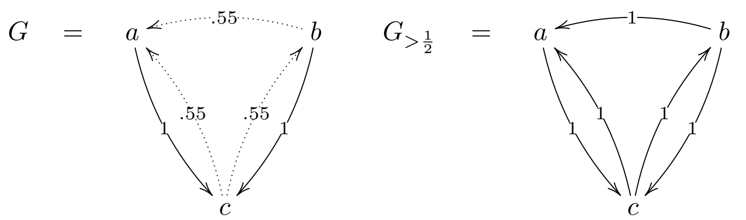
Theoretical computing complexity of the first step is in $\mathcal{O}(3^m)$, m being the dimension of the underlying finite set A. This exponential complexity must be considered w.r.t. the indeterministic efficiency of the dynamic propagation of the min and max operators by the finite domain solver we use (cf. Bisdorff & Roubens [2]). For a rather large class of practical examples this step is rather quick, as may be seen in table 1 below where median execution time is about 9 seconds for a 20 nodes graph. That the problem is NP appears with the rapid deterioration of execution times (nearly 5 minutes) for a small subset of worst cases for such graphs as shown in the last row of table 1. Step 2 has a polynomial computing complexity $\mathcal{O}(nm^3)$, where n is the kernel dimension of R. For $\mathcal{L}$-connected graphs (cf. Bisdorff & Roubens [1]), this dimension is bound by the dimension of the set A, so that it may be approximated in the worst case by $\mathcal{O}(m^4)$. Finally, step 3 is again, for the worst case, in polynomial complexity $\mathcal{O}(nm^4)$.

Nodes	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
8	0.6	0.7	0.8	0.8	0.8	1.6
10	1.1	1.2	1.4	1.5	1.5	4.4
15	3.2	3.5	3.8	5.5	5.0	48.9
20	7.4	8.2	8.8	59.2	16.1	1774.0

Table 1: CPU times (secs) for computing median β-cut kernel solutions for random $\mathcal{L}$-valued graphs

Example 4. We may reconsider the introductory example and illustrate the output of the different steps of our algorithm.

step 1 The median β-cut gives following $\mathcal{B}$-valued graph $G_{>\frac{1}{2}}$:



[4] The dominated or absorbent kernel is obtained by replacing the reversed relation R^{-1} by the original relation R (Bisdorff & Roubens [1], [2]).

The kernel solutions for this crisp relation are the following:
$K(R_{>\frac{1}{2}}) = \{[0,1,0],[0,0,1]\}$ and R has kernel-dimension[5] 2.

step 2 From these kernel solutions we may construct the $R_{/K_i}$-restrictions already mentioned in the introductory section.

step 3 Finally the dual fix-point algorithm, computing in an integer percents domain, will give, $\{a(.45), b(1), c(0)\}$ for the K_1-restriction and $\{a(.45), b(.45), c(.55)\}$ for the K_2-restriction, so that we recover the original set $K(R)$ of $\mathcal{L}$-valued kernel solutions as expected.

5 CONCLUSION

We have defined in this report an "additive" decomposition of a given $\mathcal{L}$-valued simple graph into a set of independent $\mathcal{L}$-alignments. The original relation may be naturally re-composed in a global relation of same shape and of same kernel solutions as the original graph. This interesting additive de- and recomposition may be used to implement on the basis of the median β-cut kernels, a fast algorithm for computing the corresponding $\mathcal{L}$-valued kernel solutions. Practical experiments have shown a very significant amelioration (1 to 50) in time for solving even small-sized examples as the well known car selection data of the Electre IS method (Roy & Bouyssou [6]).

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