On ranking-by-choosing with bipolar outranking digraphs of large orders

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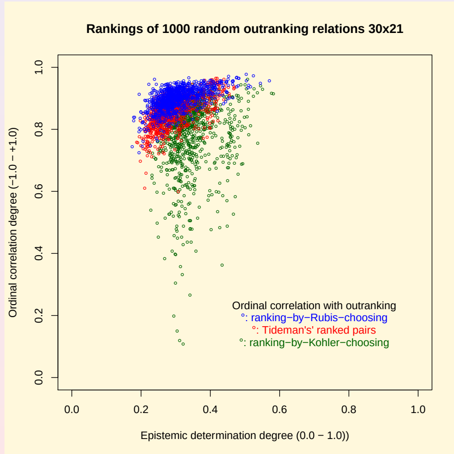
Motivation

Compared to other ranking-by-choosing rules like

- Kohler's rule,
- Arrow-Raynaud's rule (codual of Kohler's),
- Tideman's Ranked Pairs,
- Dias-Lamboray's leximin (codual of ranked pairs),

the ranking-by-Rubis-choosing rule delivers (partial) weak orderings that are most ordinally correlated with the corresponding pairwise strict outranking relation.

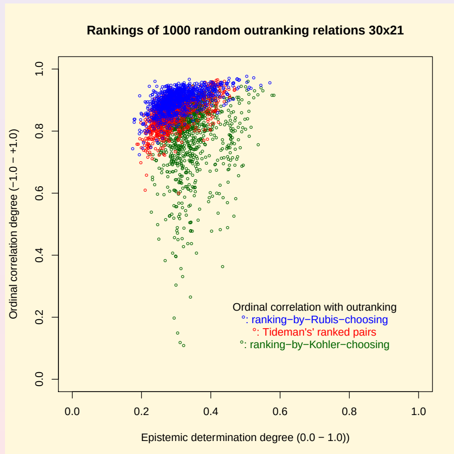


Complexity issues

- Ranking-by-Rubis-choosing consists in recursively extracting the most outranking (best) or most outranked (worst) independent choices –outranking and outranked kernels– from the remaining outranking digraph;
- Now, enumerating all kernels in a digraph becomes a computationally hard problem with large and/or sparse digraphs.
- A ranking-by-Rubis-choosing problem can, hence, only be solved for tiny digraph orders; generally less than 50 alternatives.

Complexity issues

- Similarly, Tideman's Ranked Pairs rule, due to its back-tracking strategy, cannot handle outranking digraphs showing a lot of circuits.
- Only Kohler's rule rule, being of $\mathcal{O}(n^2)$ complexity wrt to a digraph order n, can handle larger ranking problems.
- However, the quality of the Kohler ranking is not satisfactory in many cases.



Outline

In this lecture we present a two-stages decomposition of large outranking digraphs:

1. All alternatives are, first, sorted into a prefixed set of q multiple criteria quantile classes.
2. Each resulting quantile equivalence class is then locally ranked-by-choosing on the basis of the restricted outranking digraph.

This strategy allows us to potentially solve such ranking-by-choosing problems in parallel from outranking digraph of up to several thousand of decision alternatives.

Content

Performance Quantiles

- Let X be the set of n potential decision alternatives evaluated on a single real performance criteria.
- We denote $x, y, ...$ the performances observed of the potential decision actions in X.
- We call quantile $q(p)$ the performance such that $p\%$ of the observed n performances in X are less or equal to $q(p)$.
- The quantile $q(p)$ is estimated by linear interpolation from the cumulative distribution of the performances in X.

Performance Quantile Classes

- We consider a series: $p_k = k/q$ for $k = 0, ...q$ of $q + 1$ equally spaced quantiles like
  - quartiles: $0, .25, .5, .75, 1,$
  - quintiles: $0, .2, .4, .6, .8, 1,$
  - deciles: $0, .1, .2, ..., .9, 1,$ etc
- The upper-closed q^k class corresponds to the interval $]q(p_{k-1}); q(p_k)]$, for $k = 2, ..., q$, where $q(p_q) = \max_X x)$ and the first class gathers all data below p_1: $]-\infty; q(p_1)]$.
- The lower-closed q_k class corresponds to the interval $[q(p_{k-1}); q(p_k)[$,for $k = 1, ..., q-1$, where $q(p_0) = \min_X x$ and the last class gathers all data above $q(p_{q-1})$: $[q(p_{q-1}), +\infty[$.
- We call q-tiles a complete series of $k = 1, ..., q$ upper-closed q^k, resp. lower-closed q_k, quantile classes.

Example

Let us consider the following 31 random performances:

1.10	6.93	8.59	20.97	22.16	24.18	25.39	27.13
32.10	32.23	33.53	34.59	38.65	41.41	41.89	44.87
45.03	50.72	50.96	54.43	58.53	59.82	61.68	62.48
64.82	65.65	71.99	80.73	87.84	87.89	91.56	-

measured on a real scale from 0.0 to 100.0.

5-tiles class limits:

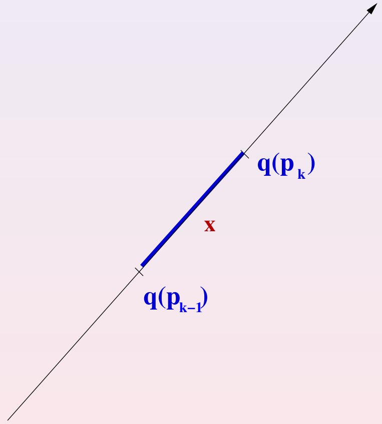
k	p_k	$[q(p_k), \_[$	$]\_, q(p_k)]$
0	0.0	1.10	$-\infty$
1	0.2	26.09	26.09
2	0.4	40.86	40.86
3	0.6	55.25	55.25
4	0.8	69.45	69.45
5	1.0	$+\infty$	91.56

5-tiles class contents:

q_k class	q^k class	#
$[0.8; +\infty[$	$]0.8; 1.0]$	5
$[0.6; 0.8[$	$]0.6; 0.8]$	6
$[0.4; 0.6[$	$]0.4; 0.6]$	7
$[0.2; 0.4[$	$]0.2; 0.4]$	6
$[0.0; 0.2[$	$]-\infty; 0.2]$	7

q-tiles sorting on a single criteria

If x is a measured performance, we may distinguish three sorting situations:



1. $x \leqslant q(p_{k-1})$ and $x < q(p_k)$
   The performance x is lower than the q^k class;
2. $x > q(p_{k-1})$ and $x \leqslant q(p_k)$
   The performance x belongs to the q^k class;
3. $(x > q(p_{k-1})$ and$)$
   $x > q(p_k)$
   The performance x is higher than the p^k class.

If the relation $<$ is the dual of $\geqslant$, it will be sufficient to check that both, $q(p_{k-1}) \not\geqslant x$, as well as $q(p_k) \geqslant x$, are verified for x to be a member of the k-th q-tiles class.

Taking into account imprecise evaluations

Example (5-tiles sorting ...)

1.1	6.9	8.6	21.0	22.2	24.2	25.4	27.1
32.1	32.2	33.5	34.6	38.6	41.4	41.9	44.9
45.0	50.7	51.0	54.4	58.5	59.8	61.7	62.5
64.8	65.7	72.0	80.7	87.8	87.9	91.6	-

Suppose now we acknowledge two preference discrimination thresholds:

1. An indifference threshold *ind* of 10.0 pts, modelling the maximal numerical performance difference which is considered preferentially insignificant;
2. A preference threshold *pr* of 20.0 pts ($pr > ind$), modelling the smallest numerical performance which is considered preferentially significant.

Resulting 5-tiles sorting:

q-tiles class	values
$]0.0 - 0.2]$	$\{1.1, 6.9, 8.6\}$
$]0.0 - 0.4]$	$\{21.0, 22.2, 24.2, 25.4\}$
$]0.2 - 0.4]$	$\{27.1\}$
$]0.2 - 0.6]$	$\{32.1, 32.2, 33.5, 34.6, 38.6\}$
$]0.4 - 0.6]$	$\{41.4, 41.9, 44.9, 45.0\}$
$]0.4 - 0.8]$	$\{50.7, 51.0, 54.4\}$
$]0.6 - 0.8]$	$\{58.5\}$
$]0.6 - 1.0]$	$\{59.8, 61.7, 62.5, 64.8, 65.7\}$
$]0.8 - 1.0]$	$\{72.0, 80.7, 87.8, 87.9, 91.6\}$

Multiple criteria extension

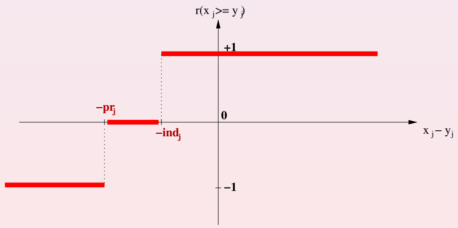
- $A = \{x, y, z, ...\}$ is a finite set of n objects to be sorted.
- $F = \{1, ..., m\}$ is a finite and coherent family of m performance criteria.
- For each criterion j in F, the objects are evaluated on a real performance scale $[0; M_j]$,
  supporting an indifference threshold ind_j
  and a preference threshold pr_j such that $0 \leqslant ind_j < pr_j \leqslant M_j$.
- The performance of object x on criterion j is denoted x_j.
- Each criterion j in F carries a rational significance w_j such that $0 < w_j < 1.0$ and $\sum_{j \in F} w_j = 1.0$.

Performing marginally *at least as good as*

Each criterion j is characterizing a double threshold order $\geqslant_j$ on A in the following way:

$$r(x \geqslant_j y) = \begin{cases} +1 & \text{if} \quad x_j - y_j \geqslant -ind_j \\ -1 & \text{if} \quad x_j - y_j \leqslant -pr_j \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

- $+1$ signifies x is *performing at least as good as* y on criterion j,
- -1 signifies that x is *not performing at least as good as* y on criterion j.
- 0 signifies that it is *unclear* whether, on criterion j, x is performing at least as good as y.



Performing globally *at least as good as*

Each criterion j contributes the significance w_j of his "*at least as good as*" characterization $r(\geqslant_j)$ to the global characterization $r(\geqslant)$ in the following way:

$$r(x \geqslant y) \; = \; \sum_{j \in F} \left[\, w_j \cdot r(x \geqslant_j y) \, \right] \quad (2)$$

- $r > 0$ signifies x is *globally performing at least as good as* y,
- $r < 0$ signifies that x is *not globally performing at least as good as* y,
- $r = 0$ signifies that it is *unclear* whether x is globally performing at least as good as y.

Performing marginally and globally *less than*

Each criterion j is characterizing a double threshold order $<_j$ (*less than*) on A in the following way:

$$r(x <_j y) \quad = \quad \begin{cases} +1 & \text{if} \quad x_j + pr_j \leqslant y_j \\ -1 & \text{if} \quad x_j + ind_j \geqslant y_j \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

And, the *global less than* relation ($<$) is defined as follows:

$$r(x < y) = \sum_{j \in F} \left[w_j \cdot r(x <_j y) \right] \quad (4)$$

Proposition

The global "less than" relation $<$ is the dual ($\not\geqslant$) of the global "at least as good as" relation $\geqslant$.

First result

Let $\mathbf{q}(p_{k-1}) = \big(q_1(p_{k-1}), q_2(p_{k-1}), ..., q_m(p_{k-1})\big)$ denote the lower limits and $\mathbf{q}(p_k) = \big(q_1(p_k), q_2(p_k), ..., q_m(p_k)\big)$ the corresponding upper limits of the q^k class on the m criteria.

Proposition

That object x belongs to class q^k,i.e. the k-th upper-closed q-tiles class $]p_{k-1}; p_k]$ $(k = 1, ..., q)$, resp. q_k, may be characterized as follows:

$$r(x \in q^k) \; = \; \min\big(\, r(\mathbf{q}(p_{k-1}) \not\geqslant x), r(\mathbf{q}(p_k) \geqslant x) \, \big)$$

$$r(x \in q_k) \; = \; \min\big(\, r(x \geqslant \mathbf{q}(p_{k-1})), r(x \not\geqslant \mathbf{q}(p_k)) \, \big)$$

Marginal *considerably better* or *worse performing* situations

On a criterion j, we characterize a *considerably less performing* situation, called veto and denoted $\lll_j$, as follows:

$$r(x \lll_j y) = \begin{cases} +1 & \text{if} \quad x_j + v_j \leqslant y_j \\ -1 & \text{if} \quad x_j - v_j \geqslant y_j \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

where v_j represents a veto discrimination threshold. A corresponding dual *considerably better performing* situation, called counter-veto and denoted $\ggg_j$, is similarly characterized as:

$$r(x \ggg_j y) = \begin{cases} +1 & \text{if} \quad x_j - v_j \geqslant y_j \\ -1 & \text{if} \quad x_j + v_j \leqslant y_j \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

Global considerably *better* or *worse performing* situations

A global *veto*, or *counter-veto* situation is now defines as follows:

$$r(x \lll y) = \circledvee_{j \in F} r(x \lll_j y) \quad (7)$$
$$r(x \ggg y) = \circledvee_{j \in F} r(x \ggg_j y) \quad (8)$$

where $\circledvee$ represents the epistemic polarising (Bisdorff 1997) or symmetric maximum (Grabisch et al. 2009) operator:

$$r \circledvee r' = \begin{cases} \max(r, r') & \text{if} \quad r \geqslant 0 \wedge r' \geqslant 0, \\ \min(r, r') & \text{if} \quad r \leqslant 0 \wedge r' \leqslant 0, \\ 0 & \text{otherwise.} \end{cases} \quad (9)$$

Characterizing veto and counter-veto situations

1. $r(x \lll y) = 1$ iff there exists a criterion j such that $r(x \lll_j y) = 1$ and there does not exist otherwise any criterion k such that $r(x \ggg_k y) = 1$.
2. Conversely, $r(x \ggg y) = 1$ iff there exists a criterion j such that $r(x \ggg_j y) = 1$ and there does not exist otherwise any criterion k such that $r(x \lll_k y) = 1$.
3. $r(x \ggg y) = 0$ if either we observe no very large performance differences or we observe at the same time, both a very large positive and a very large negative performance difference.

Lemma

$r(\not\lll)^{-1}$ is identical to $r(\ggg)$.

The bipolar outranking relation $\succsim$

From an epistemic point of view, we say that:

1. object x *outranks* object y, denoted $(x \succsim y)$, if
   - 1.1 a significant majority of criteria validates a global outranking situation between x and y, and
   - 1.2 no veto is observed on a discordant criterion,
2. object x *does not outrank* object y, denoted $(x \not\succsim y)$, if
   - 2.1 a significant majority of criteria invalidates a global outranking situation between x and y, and
   - 2.2 no counter-veto is observed on a concordant criterion.

Polarising the global "*at least as good as*" characteristic

The bipolarly-valued outranking characteristic $r(\succsim)$ is defined as follows:

$$r(x \succsim y) = \begin{cases} 0, & \text{if } \big[\exists j \in F : r(x \lll_j y)\big] \wedge \big[\exists k \in F : r(x \ggg_k y)\big] \\ \big[\, r(x \geqslant y) \circledvee -r(x \lll y) \,\big] & \text{, otherwise.} \end{cases}$$

And in particular,

- $r(x \succsim y) = r(x \geqslant y)$ if no very large positive or negative performance differences are observed,
- $r(x \succsim y) = 1$ if $r(x \geqslant y) \geqslant 0$ and $r(x \ggg y) = 1$,
- $r(x \succsim y) = -1$ if $r(x \geqslant y) \leqslant 0$ and $r(x \lll y) = 1$,

q-tiles sorting with bipolar outrankings

Proposition

The bipolar characteristic of x belonging to upper-closed q-tiles class q^k, resp. lower-closed class q_k, may hence, in a multiple criteria outranking approach, be assessed as follows:

$$r(x \in q^k) = \min\big[\, -r\big(\mathbf{q}(p_{k-1}) \succsim x\big),\ r\big(\mathbf{q}(p_k) \succsim x\big)\,\big]$$
$$r(x \in q_k) = \min\big[\, r\big(x \succsim \mathbf{q}(p_{k-1})\big),\ -r\big(x \succsim \mathbf{q}(p_k)\big)\,\big]$$

Proof.

The bipolar outranking relation $\succsim$, being weakly complete, verifies the coduality principle (Bisdorff 2013). The dual ($\not\succsim$) of $\succsim$ is, hence, identical to the strict converse outranking $\precnsim$ relation. □

The multicriteria (upper-closed) q-tiles sorting algorithm

1. **Input**: a set X of n objects with a performance table on a family of m criteria and a set $\mathcal{Q}$ of $k = 1, .., q$ empty q-tiles equivalence classes.
2. **For each** object $x \in X$ **and each** q-tiles class $q^k \in \mathcal{Q}$
   2.1 $r(x \in q^k) \quad \leftarrow \quad \min\big(-r(\mathbf{q}(p_{k-1}) \succsim x), r(\mathbf{q}(p_k) \succsim x)\big)$
   2.2 if $r(x \in q^k) \geqslant 0$:
   add x to q-tiles class q^k
3. **Output**: $\mathcal{Q}$

Comment

1. *The complexity of the q-tiles sorting algorithm is $\mathcal{O}(nmq)$; linear in the number of decision actions (n), criteria (m) and quantile classes (q).*
2. *As $\mathcal{Q}$ represents a partition of the criteria measurement scales, i.e. the upper limits of the preceding category correspond to the lower limits of the succeeding ones, there is a potential for run time optimization.*

49-tiles sorting of THE University Rankings

- THE 2010 Ranking of 34 top European Universities;
- Five cardinal criteria (measured as z-scores) for evaluating the performance of each university:
  1. Teaching: the learning environment ($w_T = 3$),
  2. Citations: research influence ($w_C = 3$),
  3. Research: volume, income and reputation ($w_R = 1$),
  4. International outlook ($w_I = 1$),
  5. Industry income: innovation ($w_{Ind} = 1$).
- Browsing the 49-tiles sorting result.

Properties of q-tiles sorting result

1. *Coherence*: Each object is always sorted into a non-empty subset of adjacent q-tiles classes.
2. *Uniqueness*: If the q-tiles classes represent a discriminated partition of the measurement scales on each criterion and $r \neq 0$, then every object is sorted into exactly one q-tiles class.
3. *Independence*: The sorting result for object x, is independent of the other object's sorting results.

Comment

The independence property gives us access to efficient parallel processing of class membership characteristics $r(x \in q^k)$ for all $x \in X$ and q^k in $\mathcal{Q}$.

The 17-tiles sorting of the THE University ranking data

]0.94 - 1.00]: {}
]0.88 - 0.94]: {}
]0.82 - 0.88]: {'ICL-UK'}
]0.76 - 0.82]: {'ETHZ-CH', 'UC-UK', 'UO-UK'}
]0.71 - 0.76]: {'ENSP-FR', 'EUT-NL', 'KI-S', 'KUL-BE', 'UC-UK', 'UCL-UK'}
]0.65 - 0.71]: {'ENSP-FR', 'EUT-NL', 'KI-S', 'KUL-BE', 'UCL-UK'}
]0.59 - 0.65]: {'EUT-NL','KI-S', 'KUL-BE', 'UCL-UK'}
]0.53 - 0.59]: {'EUT-NL', 'KI-S', 'KUL-BE', 'UCL-UK', 'UE-UK'}
]0.47 - 0.53]: {'EP-FR', 'EUT-NL', 'KI-S', 'KUL-BE', 'LSE-UK', 'UE-UK', 'UG-DE'}
]0.41 - 0.47]: {'EPFL-CH', 'EUT-NL', 'KI-S', 'KUL-BE', 'LSE-UK', 'UCD-IR', 'UE-UK', 'UG-DE', 'UM-DE', 'UM-UK', 'UZ-CH'}
]0.35 - 0.41]: {'EUT-NL', 'KI-S', 'UCD-IR', 'UM-DE', 'UM-UK'}
]0.29 - 0.35]: {'EUT-NL', 'KI-S', 'UB-UK', 'UCD-IR'}
]0.24 - 0.29]: {'ENSL-FR', 'KI-S', 'UB-CH', 'UB-UK', 'UCD-IR'}
]0.18 - 0.24]: {'DU-UK', 'ENSL-FR', 'KCL-UK', 'KI-S', 'RKU-DE', 'TUM-DE', 'UG-CH', 'UH-FI', 'USTA-UK', 'USth-UK', 'UY-UK'}
]0.12 - 0.18]: {'DU-UK', 'ENSL-FR', 'KI-S', 'TCD-IR', 'TUM-DE', 'UG-CH', 'USTA-UK'}
]0.06 - 0.12]: {'DU-UK', 'KI-S', 'LU-S', 'RHL-UK', 'UG-CH', 'US-UK'}
]< - 0.06]: {'RHL-UK'}

The 17-tiles partition

quantile class	content	quantile class	content
]0.82 - 0.88]	ICL-UK	]0.24 - 0.47]	UCD-IR
]0.76 - 0.82]	UO-UK	]0.24 - 0.35]	UB-UK
	ETHZ-CH	]0.24 - 0.29]	UB-CH
]0.71 - 0.82]	UC-UK	]0.12 - 0.29]	ENSL-FR
]0.65 - 0.76]	ENSP-FR	]0.18 - 0.24]	KCL-UK
]0.53 - 0.76]	UCL-UK		RKU-DE
]0.41 - 0.76]	KUL-BE		UY-UK
]0.29 - 0.76]	EUT-NL		UH-FI
]0.06 - 0.76]	KI-S		USth-UK
]0.41 - 0.59]	UE-UK	]0.12 - 0.24]	TUM-DE
]0.47 - 0.53]	EP-FR		USTA-UK
	LSE-UK	]0.06 - 0.24]	UG-CH
]0.41 - 0.53]	UG-DE		DU-UK
]0.41 - 0.47]	EPFL-CH	]0.12 - 0.18]	TCD-IR
	UZ-CH	]0.06 - 0.12]	US-UK
]0.35 - 0.47]	UM-DE		LU-S
	UM-UK	$]-\infty$ - 0.12]	RHL-UK

Ordering the q-tiles sorting result

The q-tiles sorting result leaves us with a more or less refined partition of the set X of n potential decision actions.

In the upper-closed 17-tiles sorting of the 2010 THE University ranking data, we obtain 23 quantile classes, of which 8 contain more than 1 action (1×5 and 7×2 actions).

For linearly ranking from best to worst the resulting parts of the q-tiles partition we may apply three strategies:

1. Optimistic: In decreasing lexicographic order of the upper and lower quantile class limits;
2. Pessimistic: In decreasing lexicographic order of the lower and upper quantile class limits;
3. Average: In decreasing numeric order of the average of the lower and upper quantile limits.

q-tiles ranking algorithm

1. **Input**: the outranking digraph $\mathcal{G}(X, \succsim)$, a partition P_q of k linearly ordered decreasing parts of X obtained by the q-sorting algorithm, and an empty list $\mathcal{R}$.
2. **For each** quantile class $q^k \in P_q$:

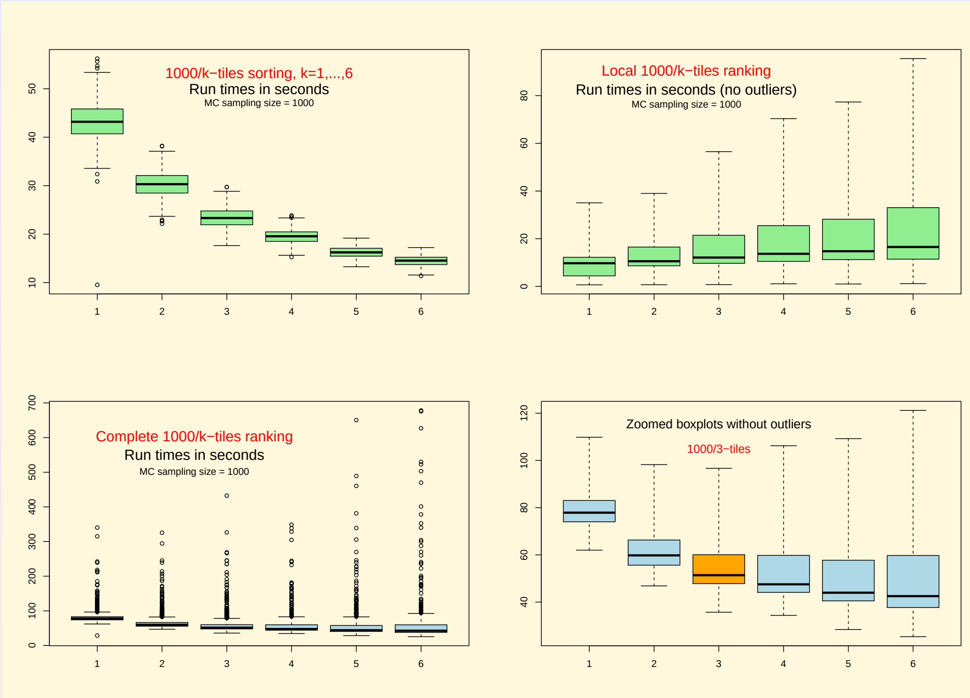
   if $\#(q^k) > 1$:
   $R_k \leftarrow$ **rank-by-choosing** q^k in $\mathcal{G}_{|q^k}$
   (if ties, render alphabetic order of action keys)
   else: $R_k \leftarrow q^k$
   append R_k to $\mathcal{R}$
3. **Output**: $\mathcal{R}$

q-tiles ranking algorithm – Comments

1. In case of local ties (very similar evaluations for instance), the **rank-by-choosing** procedure will render these actions in increasing alphabetic ordering of the action keys.
2. The complexity of the q-tiles ranking algorithm is linear in the number of parts resulting from a q-tiles sorting which contain more than one action.
3. However, the **rank-by-Rubis-choosing** procedure is NP-hard. No solution in reasonable time can be guaranteed with more than 40 decision actions.
4. In case of a larger quantile class q^k (many very similar evaluations, or many indeterminate outrankings), we may replace the rank-by-choosing procedure with a local polynomial ranking rule, like Kohler's rule or the principal projection of the covariance of the $r(\succsim)$ credibilities.

Profiling the q-tiles sorting & ranking procedure

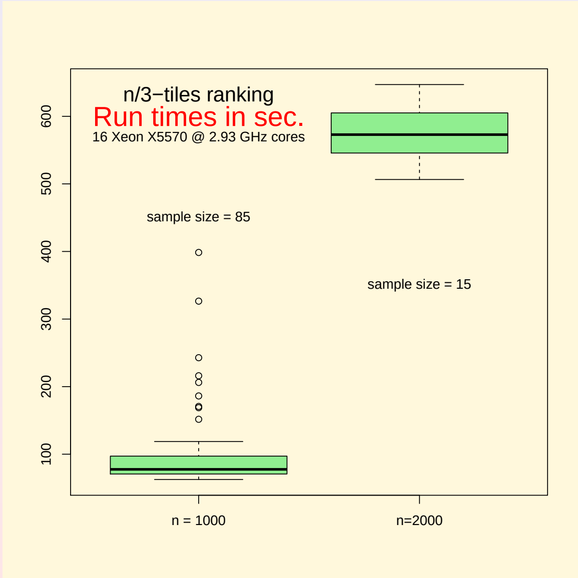
1. Due to the potential complexity of the local rank-by-Rubis-choosing procedure, the number q of sorting quantiles must be chosen with care in order that the restricted outranking digraphs $\mathcal{G}_{|q^k}$ keep tiny or small orders (<40 actions).
2. Monte Carlo experimentation with random outranking digraphs of order $n = 1000$ have shown that it is opportune to set $q = n/3$ when n gets large.



Profiling the q-tiles sorting & ranking procedure

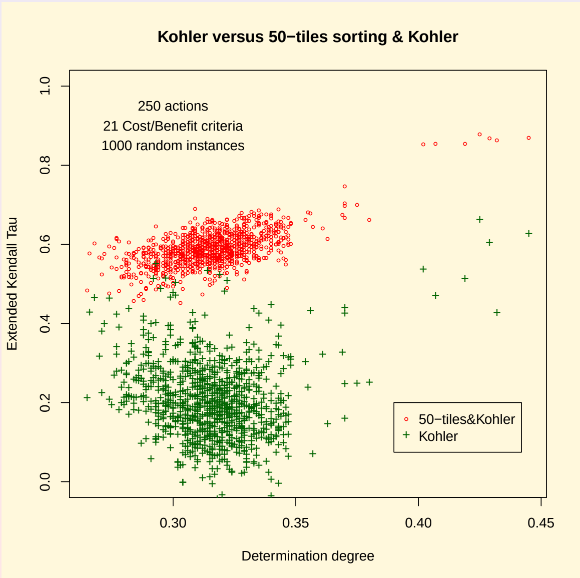
1. Following from the independence property of the q-**tiles sorting** of each action into each q-tiles class, the q-sorting algorithm may be safely split into as much threads as are multiple processing cores available in parallel.
2. Furthermore, the **rank-by-choosing** procedure being local, this procedures may thus be safely processed in parallel threads on each restricted outranking digraph $\mathcal{G}_{|q^k}$.

Multiple threading with 16 cores



Profiling the local ranking procedure

For very large orders it is opportune to use Kohler's rule for the local ranking step.



Concluding ...

- We implement a new ranking (actually: thinly weak-ordering) algorithm based on quantiles sorting and local ranking procedures;
- Final ranking result generally fits well with the underlying outranking relation;
- Independent sorting and local ranking procedures allow effective multiprocessing strategies;
- Efficient scalability allows hence the ranking of very large sets of potential decision actions (thousands of nodes) graded on multiple incommensurable criteria;
- Good perspectives for further optimization and ad-hoc fine-tuning.