Polarizing outranking relations with large performance differences

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Who wins the election?

Working hypothesis:

1. Each voter ranks without ties the potential candidates from his best to his worst candidate and communicates without cheating this ranking.
2. The election result is computed by aggregating directly these marginal rankings into a global compromise one.

Comment

Two seminal aggregation methods, quite different in their spirit, have been proposed in the 18th century by two French scientists:

Marie Jean Antoine Nicolas Caritat, marquis de Condorcet (17 septembre 1743 – 28 mars 1794) mathématicien, philosophe et politologue.

Jean-Charles Chevalier de Borda (4 mai 1733 – 19 février 1799) ingénieur du génie maritime, mathématicien, physicien et politologue.

Condorcet's method

Principle (Condorcet 18th century)

- In 1785, Condorcet suggests to compare pairwise all the potential candidates.
- Candidate a is preferred to candidate b if and only if the number of voters who rank a before b is higher than the number of voters who ranks b before a.
- A candidate, who is thus preferred to all the others, wins the election and is called Condorcet winner.

Comment

- *The Condorcet winner is always preferred by a majority of voters to all the other candidates.*
- *He always defeats all the other candidates in a sequential election.*
- *A Condorcet winner is always unique.*

Condorcet's Approach

Example (*Evaluation and Decision Models, Bouyssou et al, Kluwer 2000 p.14*)

- Let $\{a, b, c, d, e, f, g, \mathbf{x}, \mathbf{y}\}$ be the set of candidates in a 101 voters election. Suppose that:
  - 19 voters have preferences $\mathbf{y}abcdefg\mathbf{x}$,
  - 21 voters have preferences $efg\mathbf{x}\mathbf{y}abcd$,
  - 10 voters have preferences $e\mathbf{x}\mathbf{y}abcdfg$,
  - 10 voters have preferences $f\mathbf{x}\mathbf{y}abcdeg$,
  - 10 voters have preferences $g\mathbf{x}\mathbf{y}abcdef$,and
  - 31 voters have preferences $\mathbf{y}abcd\mathbf{x}efg$.
- Candidate $\mathbf{x}$ is here the Condorcet winner.
- However, is candidate $\mathbf{x}$ *really better than* candidate $\mathbf{y}$?

The classical outranking approach

- In a multiple criteria decision aid context, the candidates are the decision alternatives, the voters are the criteria and the votes are described by a performance tableau.
- Here we say that a decision alternative x outranks a decision alternative y if:
  1. a potentially weighted majority of criteria validates the statement that x *performs at least as good as* y and
  2. **no** considerably *large negative marginal performance difference* is observed in disfavour of x.

Comment

In the previous election, the statement that candidate x is preferred to candidate y by a majority of voters, would be put to doubt by the considerably large negative performance difference (last ranked versus first ranked) observed in disfavour of x.

Notations

- $A = \{x, y, z, ...\}$ is a finite set of decision alternatives;
- $F = \{1, ..., n\}$ is a finite and coherent family of performance criteria;
- For each criterion i in F, the alternatives are evaluated on a real performance scale $[0; M_i]$,

  supporting an indifference threshold q_i,

  and a preference threshold p_i such that $0 \leqslant q_i < p_i \leqslant M_i$;
- The performance of alternative x on criterion i is denoted x_i;
- Each criterion i in F carries a rational significance w_i such that $0 < w_i < 1.0$ and $\sum_{i \in F} w_i = 1.0$.

Performing marginally *at least as good as*

Each criterion i is characterising a double threshold order $\geqslant_i$ on A in the following way:

$$r(x \geqslant_i y) = \begin{cases} +1 & \text{if} \quad x_i + q_i \geqslant y_i \\ -1 & \text{if} \quad x_i + p_i \leqslant y_i \\ 0 & \text{otherwise.} \end{cases}$$

$+1$ signifies x is *performing at least as good as* y on criterion i,

-1 signifies that x is *not performing at least as good as* y on criterion i.

0 signifies that it is *unclear* whether, on criterion i, x is performing at least as good as y.

"*At least as good as*" Majority Margins

Each criterion i contributes the significance w_i of his "*at least as good as*" characterisation $r(\geqslant_i)$ to the global characterisation $r(\geqslant)$ in the following way:

$$r(x \geqslant y) \;=\; \sum_{i \in F} \left[\, w_i \cdot r(x \geqslant_i y) \,\right]$$

$r > 0$ signifies x is *globally performing at least as good as* y,

$r < 0$ signifies that x is *not globally performing at least as good as* y,

$r = 0$ signifies that it is *unclear* whether x is globally performing at least as good as y.

Properties of the "*at least as good as*" majority margins

1. The "*performing at least as good as*" majority relation is reflexive: $r(x \geqslant x) = +1$ for all x in A (Bouyssou 1996)
2. The "*performing at least as good as*" majority relation is weakly complete: $r(x \geqslant y) < 0$ implies that $r(y \geqslant x) \geq 0$, for all x, y in A.
3. The asymmetric part $>$ ("*better than*" part) of the relation $\geqslant$ corresponds to its codual ($\not\geqslant^{-1}$) relation.

Marginal *considerably better* or *worse performing* situations

We redefine a single threshold order, denoted $\lll_i$ which represents *considerably less performing* situations as follows:

$$r(x \lll_i y) = \begin{cases} +1 & \text{if} \quad x_i + v_i \leqslant y_i \\ -1 & \text{if} \quad x_i - v_i \geqslant y_i \\ 0 & \text{otherwise.} \end{cases} .$$

And a corresponding dual *considerably better performing* situation $\ggg_i$ characterised as:

$$r(x \ggg_i y) = \begin{cases} +1 & \text{if} \quad x_i - v_i \geqslant y_i \\ -1 & \text{if} \quad x_i + v_i \leqslant y_i \\ 0 & \text{otherwise.} \end{cases} . \qquad (1)$$

Global considerably *better* or *worse performing* situations

A global *veto*, or *counter-veto* situation is now defines as follows:

$$r(x \lll y) = \bigcirc\!\!\!\!\vee_{i \in F}\, r(x \lll_i y) \qquad (2)$$

$$r(x \ggg y) = \bigcirc\!\!\!\!\vee_{i \in F}\, r(x \ggg_i y) \qquad (3)$$

where $\bigcirc\!\!\!\!\vee$ represents the epistemic polarizing (Bisdorff 1997) or symmetric maximum (Grabisch et al. 2009) operator:

$$r \,\bigcirc\!\!\!\!\vee\, r' = \begin{cases} \max(r, r') & \text{if} \quad r \geqslant 0 \wedge r' \geqslant 0, \\ \min(r, r') & \text{if} \quad r \leqslant 0 \wedge r' \leqslant 0, \\ 0 & \text{otherwise.} \end{cases} \qquad (4)$$

Comment

!!! The $\bigcirc\!\!\!\!\vee$ operator not being associative, we render its result unambiguous by first gathering separately the positive terms and the negative terms and then only applying the computation rule.

Characterising veto and counter-veto situations

1. $r(x \lll y) = 1$ iff there exists a criterion i such that $r(x \lll_i y) = 1$ and there does not exist otherwise any criteria j such that $r(x \ggg_j y) = 1$.
2. Conversely, $r(x \ggg y) = 1$ iff there exists a criterion i such that $r(x \ggg_i y) = 1$ and there does not exist otherwise any criteria j such that $r(x \lll_j y) = 1$.
3. $r(x \ggg y) = 0$ if either we observe no very large performance differences or we observe at the same time, both a very large positive and a very large negative performance difference.

Property (Self-coduality)

$r(\not\lll)^{-1}$ is identical to $r(\ggg)$.

The polarized outranking relation $\succsim$

From an epistemic point of view, we say that:

1. alternative x *outranks* alternative y, denoted ($x \succsim y$), if
   1.1 a significant majority of criteria validates a global outranking situation between x and y, and
   1.2 no serious counter-performance is observed on a discordant criterion,
2. alternative x *does not outrank* alternative y, denoted ($x \not\succsim y$), if
   2.1 a significant majority of criteria invalidates a global outranking situation between x and y, and
   2.2 no considerably better performing situation is observed on a concordant criterion.

Polarizing the global "$r(\succsim)$" characteristic

The polarized bipolar-valued characteristic $r(\succsim)$ is defined as follows:

$$r(x \succsim y) = \big[\, r(x \geqslant y) \circledvee_{i \in F} r(x \lll_i y) \,\big]$$

And in particular,

- $r(x \succsim y) = r(x \geqslant y)$ if no very large positive or negative performance differences are observed,
- $r(x \succsim y) = 1$ if $r(x \geqslant y) \geqslant 0$ and $r(x \ggg y) = 1$,
- $r(x \succsim y) = -1$ if $r(x \geqslant y) \leqslant 0$ and $r(x \lll y) = 1$,
- $r(x \succsim y) = 0$ otherwise.

Properties of the polarized outranking relation

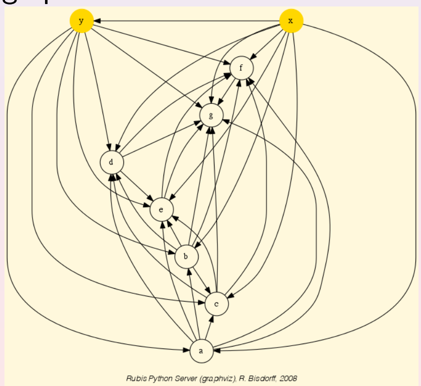
1. The "*polarized outranking*" relation is reflexive:
   $r(x \succsim x) \;=\; +1$ for all x in A
2. The "*polarized outranking*" relation is weakly complete:
   $r(x \succsim y) < 0$ implies that $r(y \succsim x) \geq 0$, for all x, y in A.
3. The asymmetric part $\succ$ of the relation $\succsim$ corresponds to its codual, i.e. the negation of its converse ($\not\succsim^{-1}$) relation.

Reconsidering the introductory example

Outranking without considering large performance differences:

Relation Table

R	a	b	c	d	e	f	g	x	y
a	0.00	1.00	1.00	1.00	0.39	0.39	0.39	-0.01	-1.00
b	-1.00	0.00	1.00	1.00	0.39	0.39	0.39	-0.01	-1.00
c	-1.00	-1.00	0.00	1.00	0.39	0.39	0.39	-0.01	-1.00
d	-1.00	-1.00	-1.00	0.00	0.39	0.39	0.39	-0.01	-1.00
e	-0.39	-0.39	-0.39	-0.39	0.00	0.80	0.80	-0.01	-0.39
f	-0.39	-0.39	-0.39	-0.39	-0.80	0.00	0.80	-0.01	-0.39
g	-0.39	-0.39	-0.39	-0.39	-0.80	-0.80	0.00	-0.01	-0.39
x	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.00	0.01
y	1.00	1.00	1.00	1.00	0.39	0.39	0.39	-0.01	0.00



Reconsidering the introductory example

Polarized outranking with large performance differences:

Relation Table

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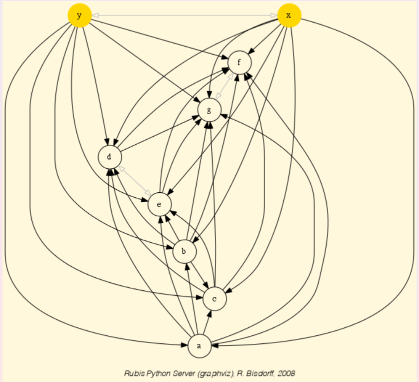
c	-1.00	-1.00	0.00	1.00	0.39	0.39	0.39	-0.01	-1.00
d	-1.00	-1.00	-1.00	0.00	0.00	0.39	0.39	-0.01	-1.00
e	-0.39	-0.39	-0.39	0.00	0.00	0.80	1.00	-0.01	-0.39
f	-0.39	-0.39	-0.39	-0.39	-0.80	0.00	0.00	-0.01	-0.39
g	-0.39	-0.39	-0.39	-0.39	-1.00	0.00	0.00	-0.01	-1.00
x	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.00	0.00
y	1.00	1.00	1.00	1.00	0.39	0.39	1.00	0.00	0.00



Concluding ...

- Outranking without taking into account large performance differences may give doubtful results;
- Polarizing the outranking with large performance differences makes apparent doubtful preference statements;
- Compared to the classical Electre outranking, the bipolarly polarized outranking shows nice properties, like weak completeness, and more important, a consistent negation of the converse relation.
- Incomparability is modelled as indeterminate preference statement. Its negation remains an indeterminate statement