The Electre like outranking approach to MCDA

I: Foundations

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11th MCDA/M Summer School 2013
Helmut-Schmidt-Universität, Hamburg, Germany

July 22nd– August 2nd, 2013

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The outranking situation

- We say that "a decision alternative *a* outranks a decision alternative *b*" if and only:
  1. There is a weighted majority of criteria, or objectives (coalitions of criteria) who warrant that *a* is perceived at least as good as *b* and,
  2. No considerable negative performance difference is observed between *a* and *b* on any criterion (or objective).
- We say that "a decision alternative *a* does not outrank a decision alternative *b*" if and only if:
  3. There is only a weighted minority of criteria, or objectives, who warrant that *a* is perceived at least as good as *b* and,
  4. No considerable positive performance difference between *a* and *b* is observed on any criterion (or objective).
- Case (2), respectively (4), is called a veto, respectively a counter–veto situation.

Best office choice (1)

Example

Let us consider the following best office choice problem.

- A SME, specialized in printing and copy services, has to move into new offices.
- The CEO of the SME has gathered the performances of seven potential office sites with respect to three objectives:

Site	Costs (↓) (in €)	Turnover (↑) (0-81 pts)	Work. Cond. (↑) (0-19 pts)
A	35 000	70.6	10.2
B	17 800	29.5	9.9
C	6 700	43.8	3.6
D	14 100	42.3	10.0
E	34.800	49.1	15.7
F	18 600	16.1	4.8
G	12 000	49.1	10.4

Best office choice (2)

Example (Significant preferential judgments)

- The CEO judges the "*Costs*" and the cumulated "*Benefits*" objectives ("*Turnover*" and "*Working Conditions*") to be equi-significant for selecting the best office site.
- Hence, the CEO considers that a concordant preferential judgment with respect to "*Costs*" and one of the two "*Benefits*" objectives is significant for him.

Certainly confirmed outranking situation

Example

Site	Costs (↓) (in €)	Turnover (↑) (0-80 pts)	Work.Cond. (↑) (0-20 pts)
G	12 000	49.1	10.4
F	18 600	16.1	4.8

- Site *G* certainly outranks site *F* as *G* is at least as well performing than *F* on all three objectives (unanimous concordance = Pareto dominance).

Positively confirmed outranking situation

Example

Site	Costs (↓) (in €)	Turnover (↑) (0-80 pts)	Work.Cond. (↑) (0-20 pts)
C	6 700	43.8	3.6
B	17 800	29.5	9.9

- Site *C* outranks site *B* as *C* is at least as well performing than *B* on objective "*Costs*" (6 700 against 17 800) and on objective "*Turnover*" (43.8 against 29.5).

Positively rejected outranking situations

Example

Site	Costs (↓) (in €)	Turnover (↑) (0-80 pts)	Work.Cond. (↑) (0-20 pts)
F	18 600	16.1	4.8
G	12 000	49.1	10.4
C	6 700	43.8	3.6

- Site *F* certainly does not outrank site *G* as *F* is less performing than *G* on all three objectives (unanimous concordance = Pareto dominance).
- Site *F* does not outrank site *C* as *F* is less performing than *C* on objective "*Costs*" (18 600 against 6 700) and objective "*Turnover*" (16.1 against 43.8).

Indeterminate outranking situation

Example

Site	Costs (↓) (in €)	Turnover (↑) (0-80 pts)	Work.Cond. (↑) (0-20 pts)
F	18 600	16.1	4.8
E	34.800	49.1	15.7

- As site *F* is less expensive than site *E* (18 600 against 34 800), but also, at the same time less advantageous on objective "*Turnover*" (16.1 against 49.1) and objective "*Work.Cond.*" (4.8 against 15.7), one can neither confirm, nor reject this outranking situation.
- This indeterminate situation is similar to a voting result where the number of favorable votes balance the number of disfavorable votes.

Indeterminate outranking situation

Example

Site	Costs (↓) (in €)	Turnover (↑) (0-80 pts)	Work.Cond. (↑) (0-20 pts)
B	17 800	29.5	9.9
A	35 000	70.6	10.2

- A similar indeterminate outranking situation is observed when comparing sites *B* and *A*. On the one hand, *B* is less expensive than site *A* (17 800 against 35 000), but, on the other hand, *B* is less advantageous both on objective "*Turnover*" (29.5 against 70.6) ans on objective "*Work.Cond.*" (9.9 against 10.2).
- Yet, is the performance difference of 0.3 pts between grades 9.9 and 10.2 effectively significant ?

Taking into account the performances' imprecision

Definition (Discrimination thresholds)

The concept of performance discrimination threshold allows to take into account on each criterion (or objective):

- The imprecision of our knowledge about present or past facts;
- The uncertainty which necessarily affects our knowledge of the future;
- The difficulties to quantify essentially qualitative consequences.

Taking into account the performances' imprecision

Definition (Discrimination thresholds – continue)

- Performance discrimination thresholds allow us to model the fact that the numerical difference observed between the performances of two potential decision alternatives on a criterion (or objective) may be:
  1. compatible with them being indifferent (indifference threshold)
  2. warranting a clear preference of one over the other (preference threshold)

Best office site for the SME

- Let us reconsider the performance table of our best office choice problem:

Site	Costs (↓) (in €)	Turnover (↑) (0-80 pts)	Work.Cond. (↑) (0-20 pts)
B	17 800 €	29.5	9.9
A	35 000 €	70.6	10.2

  A difference of 0.5 pts on objective "*Work.Cond.*" is still considered to compatible with an indifference judgment of the potential office sites,

- Hence, site B outranks site A, as the former is clearly less expensive (17 800 against 35 000) and also more or less at least as good as A on objective "*Work.Cond.*"; a 0.3 pts difference being smaller than the supposed indifference threshold.

Taking into account large performance differences

Definition (Veto situations)

- The concept of veto situation allows us to take into account on each criterion (or objective):

  the presence of a negative performance difference large enough to render insignificant the otherwise observed weighted majority of concordance of a preferential judgment.

- or, similarly:

  the presence of a positive performance difference large enough, to render insignificant the otherwise warranted weighted minority of concordance of a preferential judgment.

Taking into account large performance differences

Definition (Veto thresholds)

The concept of veto threshold allows us to model the fact that the performance difference observed between two potential decision alternatives on a criterion (or objective) may be:

either, attesting the presence of a counter-performance large enough to put to doubt a significantly affirmed outranking situation;

or, attesting the presence of an out-performance large enough to put to doubt a significantly refuted outranking situation.

Revisiting the best office site problem

- Consider the performances of alternatives A and F with respect to the three objectives:

Site	Costs (↓) (in €)	Turnover (↑) (0-80 pts)	Work.Cond. (↑) (0-20 pts)
A	35 000 €	70.6	10.2
F	18 600 €	16.1	4.8

  The outranking situation between A and F is indeterminate.

- Let the CEO consider that a performance difference of 50 pts on the "Turnover" objective attests for him a veto situation.

  Hence, the out-performance on objective "*Turnover*" of site A over site F ($70.6 - 16.1 = 54.6 > 50.0$ pts) resolves this indeterminateness in favour of site A.

  Similarly, site F does certainly not outrank site A, as the counter-performance on objective "*Turnover*" is so high that it renders insignificant the fact that F is less expensive (18600 against 35000).

Notation

- Let X be a finite set of p decision alternatives.
- Let N be a finite set of $n > 1$ criteria supporting an increasing real performance scale from 0 to M_i.
- Let $0 \leqslant q_i < p_i < v_i \leqslant M_i + \epsilon$ represent resp. the indifference, the preference, and the veto discrimination threshold observed on criterion i.
- Let $w_i \in \mathbb{Q}$ be the significance of criterion i in the global preference modelling.
- Let W be the sum of all marginal significances.
- Let x and y be two alternatives in X.
- Let x_i be the performance of x on criterion i

Performing marginally *at least as good as*

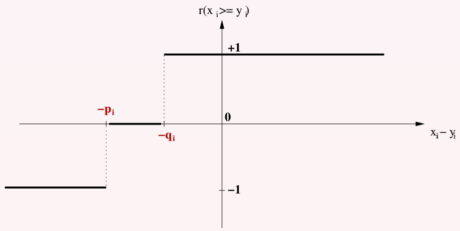
Each criterion i is characterising a double threshold order $\geqslant_i$ on A in the following way:

$$r(x \geqslant_i y) = \begin{cases} +1 & \text{if} \quad x_i + q_i \geqslant y_i \\ -1 & \text{if} \quad x_i + p_i \leqslant y_i \\ 0 & \text{otherwise.} \end{cases} \qquad (1)$$

$+1$ signifies that "x is *performing at least as good as* y" on criterion i,

-1 signifies that "x is *not performing at least as good as* y" on criterion i.

0 signifies that "it is *unclear* whether, on criterion i, x is performing at least as good as y".



Performing globally *at least as good as*

Each criterion i contributes the significance w_i of his "*at least as good as*" characterisation $r(\geqslant_i)$ to the characterisation of a global "*at least as good as*" relation $r(\geqslant)$ in the following way:

$$\boxed{r(x \geqslant y) \;=\; \sum_{i \in F} \left[\frac{w_i}{W} \cdot r(x \geqslant_i y) \right]} \qquad (2)$$

Denotational semantics:

1. $1.0 \geqslant r(x \geqslant y) > 0.0$ signifies x is *globally performing at least as good as* y,
2. $-1.0 \leqslant r(x \geqslant y) < 0.0$ signifies that x is *not globally performing at least as good as* y,
3. $r(x \geqslant y) = 0.0$ signifies that it is *unclear* whether x is globally performing at least as good as y.
4. The global "*at least as good as*" defines a median relation at minimal sum of Kendall distances from the marginal "*at least as good as*" relations.

Epistemic truth semantics of the r-valuation

Let ξ and υ be two preferential statements.

- $r(\xi) = +1$ means that assertion ξ is certainly valid,
- $r(\xi) = -1$ means that assertion ξ is certainly invalid,
- $r(\xi) > 0$ means that assertion ξ is more valid than invalid,
- $r(\xi) < 0$ means that assertion ξ is more invalid than valid,
- $r(\xi) = 0$ means that validity of assertion ξ is indeterminate,
- $r(\xi) > r(\upsilon)$ means that assertion ξ is more valid than assertion υ,
- $r(\neg \xi) = -r(\xi)$ logical (strong) negation operates by changing sign,
- $r(\xi \vee \upsilon) = \max(r(\xi), r(\upsilon))$ logical disjonction operates via the *max* operator,
- $r(\xi \wedge \upsilon) = \min(r(\xi, r(\upsilon))$ logical conjonction operates via the *min* operator.

Performing marginally and globally *less than*

Each criterion i is characterising a double threshold order $<_i$ (*less than*) on A in the following way:

$$r(x <_i y) \quad = \quad \begin{cases} +1 & \text{if} \quad x_i + p_i \leqslant y_i \\ -1 & \text{if} \quad x_i + q_i \geqslant y_i \\ 0 & \text{otherwise.} \end{cases} \qquad (3)$$

And, the *global less than* relation ($<$) is defined as follows:

$$\boxed{r(x < y) = \sum_{i \in F} \left[\frac{w_i}{W} \cdot r(x <_i y) \right]} \qquad (4)$$

Property

The global "less than" relation $<$ is the dual ($\not\geqslant$) of the global "at least as good as" relation $\geqslant$.

Taking into account vetoes

Roy introduced the concept of veto threshold v_i ($p_i < v_i \leqslant M_i + \epsilon$) to characterise the observation of *considerably less performing situations* on the family of criteria. This leads to a single threshold order, denoted $\ll_i$ which characterises considerably less performing situations as follows:

$$r(x \ll_i y) \quad = \quad \begin{cases} +1 & \text{if} \quad x_i + v_i \leqslant y_i \\ -1 & \text{otherwise} \end{cases} \qquad (5)$$

And a global veto situation $x \ll y$ is characterised as:

$$r(x \ll y) \; = \; r\Big(\bigvee_{i \in F} (x \ll_i y) \Big) = \max_{i \in F} \big[\, r(x \ll_i y) \big] \qquad (6)$$

The classic Electre outranking relation

An object x *outranks* an object y , denoted $x \succcurlyeq y$, when:

1. a *significant majority* of criteria validates the fact that x is performing at least as good as s, i.e. $r(x \geqslant y) \geq \lambda$, where λ represent a significant majority margin.
2. And, there is *no veto* raised against this claim, i.e. $r(x \ll y) < 0$.

The corresponding characteristic gives:

$$\begin{aligned} r(x \succcurlyeq y) \;&=\; r\big[(x \geqslant y) \;\wedge\; (x \not\ll y) \big] \\ &=\; \min \big[\, r(x \geqslant y), \, -r(x \ll y) \big] \end{aligned}$$

Problem with the classic Electre outranking concept

Property

Let $\succcurlyeq$ be the classic Electre outranking relation.

- *The asymmetric part $\succneqq$ of the $\succcurlyeq$, i.e. $(x \succcurlyeq y)$ and $(y \not\succcurlyeq x)$, is in general* not identical *to its codual relation $\not\preccurlyeq$ (Pirlot & Bouyssou 2009).*
- *Apart from the unanimous case, where $r(x \succcurlyeq_i y) = \pm 1.0$ for all criteria, the* absence *of any* veto *situation is sufficient and necessary for making $\succneqq\, = \,\not\preccurlyeq$.*

Comment

1. *This hiatus raises a serious concern with respect to the logical soundness of the classical Electre outranking model;*
2. *Only the complete absence of any veto mechanism can guarantee the coduality principle;*
3. *This, however, is obliterating the very interest of the outranking concept.*

Marginal *considerably better* or *worse performing* situations

We define a single threshold order, denoted $\lll_i$ which represents *considerably less performing* situations as follows:

$$r(x \lll_i y) = \begin{cases} +1 & \text{if} \quad x_i + v_i \leqslant y_i \\ -1 & \text{if} \quad x_i - v_i \geqslant y_i \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

And a corresponding dual *considerably better performing* situation $\ggg_i$ characterised as:

$$r(x \ggg_i y) = \begin{cases} +1 & \text{if} \quad x_i - v_i \geqslant y_i \\ -1 & \text{if} \quad x_i + v_i \leqslant y_i \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Global *considerably better* or *considerably worse performing* situations

A global *veto*, or *counter-veto* situation is defined as follows:

$$r(x \lll y) \;=\; \circledvee_{i \in F}\big[r(x \lll_i y)\big], \tag{9}$$
$$r(x \ggg y) \;=\; \circledvee_{i \in F}\big[r(x \ggg_i y)\big]. \tag{10}$$

where $\circledvee$ represents the epistemic polarising (Bisdorff 1997) or symmetric maximum (Grabisch et al. 2009) operator:

$$r \circledvee r' = \begin{cases} \max(r, r') & \text{if} \quad r \geqslant 0 \wedge r' \geqslant 0, \\ \min(r, r') & \text{if} \quad r \leqslant 0 \wedge r' \leqslant 0, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Characterising veto and counter-veto situations

1. $r(x \lll y) = 1$ iff there exists a criterion i such that $r(x \lll_i y) = 1$ and there does not exist otherwise any criterion j such that $r(x \ggg_j y) = 1$.
2. Conversely, $r(x \ggg y) = 1$ iff there exists a criterion i such that $r(x \ggg_i y) = 1$ and there does not exist otherwise any criterion j such that $r(x \lll_j y) = 1$.
3. $r(x \ggg y) = 0$ if either we observe no very large performance differences or we observe at the same time, both a very large positive and a very large negative performance difference.
4. $\lll$ verifies the coduality principle: $r(\not\lll)^{-1}$ is identical to $r(\ggg)$.

The bipolar outranking relation $\succsim$

From an epistemic point of view, we say that:

1. alternative x *outranks* alternative y, denoted $(x \succsim y)$, if
   1.1 a weighted majority of criteria validates a global outranking situation between x and y, and
   1.2 no considerable counter-performance is observed on a discordant criterion,
2. alternative x *does not outrank* alternative y, denoted $(x \not\succsim y)$, if
   2.1 a weighted majority of criteria invalidates a global outranking situation between x and y, and
   2.2 no considerably better performing situation is observed on a concordant criterion.

Polarising the global "*at least as good as*" characteristic

The bipolarly-valued characteristic $r(\succsim)$ is defined as follows:

$$r(x \succsim y) = r(x \geqslant y) \circledvee r(x \not\lll_1 y) \circledvee \ldots \circledvee r(x \not\lll_n y)$$

Properties:

1. $r(x \succsim y) = r(x \geqslant y)$ if no considerable positive or negative performance differences between x and y are observed;
2. $r(x \succsim y) = 1.0$ when $r(x \geqslant y) \geqslant 0$ and $r(x \ggg y) = 1.0$;
3. $r(x \succsim y) = -1.0$ when $r(x \geqslant y) \leqslant 0$ and $r(x \lll y) = 1.0$;
4. If $r(x \succsim y) < 0.0$ then $r(y \succsim x) \geqslant 0.0$.
   We say that $\succsim$ is a weakly complete on X.

Semantics of the bipolar characteristic valuation

The valuation $r(\succsim)$ has following interpretation:

- $r(\succsim(x, y)) = +1.0$ signifies that the statement $x \succsim y$ is certainly valid.
- $r(\succsim(x, y)) = -1.0$ signifies that the statement $x \succsim y$ is certainely invalid.
- $r(\succsim(x, y)) > 0$ signifies that the statement $x \succsim y$ is more valid than invalid.
- $r(\succsim(x, y)) < 0$ signifies that $x \succsim y$ is more invalid than valid.
- $r(\succsim(x, y)) = 0$ signifies that the statement $x \succsim y$ is indeterminate.

$\succsim$ verifies the coduality principle

Property

The dual $(\not\succsim)$ of the bipolarly-valued outranking relation $\succsim$ is identical to the strict converse outranking $\precnsim$ relation.

Proof: We only have to check the case where $r(x \lll y) \neq 0.0$:

$$\begin{aligned} r(x \not\succsim y) &= -r(x \succsim y) = -\big[r(x \geqslant y) \circledvee -r(x \lll y)\big] \\ &= \big[-r(x \geqslant y) \circledvee r(x \lll y)\big] \\ &= \big[r(x \not\geqslant y) \circledvee -r(x \ggg y)\big] \\ &= \big[r(x < y) \circledvee r(x \not\ggg y)\big] = r(x \precnsim y). \end{aligned}$$

Else, there exist conjointly two criteria i and j such that $r(x \lll_i y) = 1.0$ and $r(x \ggg_j y) = 1.0$. Hence, $r(x \succsim y) = r(x \not\succsim y) = r(x \precnsim y) = 0.0$. □

The bipolarly-valued outranking digraph

Definition

- We denote $\widetilde{G}(X, r(\succsim))$ the bipolarly-valued digraph modelled by $r(\succsim)$ on the set of potential decision alternatives X.
- We denote $G(X, \succsim)$, the crisp digraph associated with $\widetilde{G}$ where we retain all arcs $x \succsim y$ such that $r(x \succsim y) > 0$.
- $G(X, \succsim)$ is called the Condorcet or median cut digraph associated with $\widetilde{G}(X, r(\succsim))$.

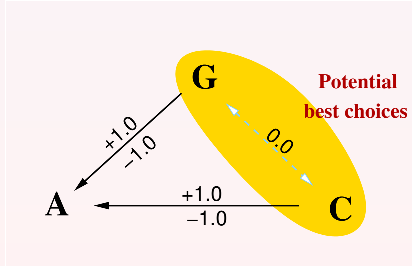
The office site choice problem revisited

If the CEO considers the following preference discrimination thresholds and significance weights:

Objective	indifference	preference	veto	weight
"Costs"	1000€	2500€	20000€	2
"Turnover"	2.5 pts	5 pts	50 pts	1
"Work. Cond."	0.5 pt	1 pt	10 pts	1

the global characteristic of the bipolar outranking relation $\succsim$ restricted, for instance, to the following three sites $\{A, C, G\}$ becomes:

$r(\succsim)$	A	C	G
A	-	-1.0	-1.0
C	+1.0	-	0.0
G	+1.0	0.0	-



The office site choice problem – continue

Site	Costs (↓)	Turnover (↑)	Work.Cond. (↑)
weight	2.0	1.0	1.0
indiff.	1000€	2.5	0.5
pref.	2500€	5	1
veto	20 000€	50	10
A	35 000	70.6	10.2
C	6 700	43.8	3.6
G	12 000	49.1	10.4

Comment

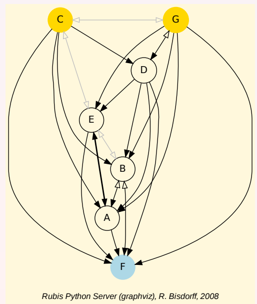
- *The bipolar outranking characteristics show that:*
  1. *Sites G and C are significantly better performing than site A.*
  2. *No significant outranking situations may be confirmed between sites G and C.*
- *Hence, both G and C may be recommended as potential best choices.*

The complete bipolar outranking relation

$r(x \succsim y)$	A	B	C	D	E	F	G
A	–	0.00	−1.00	−1.00	+0.50	+1.00	−1.00
B	+0.50	–	−0.50	−0.50	+0.00	+1.00	−0.50
C	+1.00	+0.50	–	+0.50	0.00	+0.50	0.00
D	+1.00	+1.00	−0.25	–	+1.00	+1.00	0.00
E	+0.50	0.00	0.00	−1.00	–	+1.00	−1.00
F	−1.00	0.00	−0.50	−1.00	−1.00	–	−1.00
G	+1.00	+1.00	0.00	+1.00	+1.00	+1.00	–

- No apparent Condorcet winner, but
- two weak Condorcet winners: $\{C, G\}$.
- Both together give the dominating kernel of $G(X, \succsim)$.

Site weight	Costs (↓) (2.0)	Turnover (↑) (1.0)	Work.Cond. (↑) (1.0)
C	6 700	43.8	3.6
G	12 000	49.1	10.4



RUBIS : a best choice recommender system

[1] R. Bisdorff, M. Pirlot and M. Roubens, *Choices and kernels from bipolar valued digraphs.* European Journal of Operational Research, 175 (2006) 155-170.

[2] R. Bisdorff, P. Meyer and M. Roubens, *RUBIS: a bipolar-valued outranking method for the choice problem.* 4OR, A Quarterly Journal of Operations Research, Springer-Verlag, Volume 6 Number 2 (2008) 143-165.

- Traditionally, solving a best choice problem consists in finding the unique best decision alternative.
- In RUBIS , we adopt a modern recommender system's approach which shows a subset of alternatives which contains by construction the potential best alternative(s).
- If not reduced to a singleton, the Best Choice Recommendation (BCR), has to be refined in a later decision process phase.

Pragmatic principles for a best choice recommendation

$\mathcal{P}_1$: Elimination for well motivated reasons.
Each eliminated alternative has to be outranked by at least one alternive in the BCR.

$\mathcal{P}_2$: Minimal size.
The BCR must be as limited in cardinality as possible.

$\mathcal{P}_3$: Efficient and informative.
The BCR must not contain a self-contained sub-recommendation.

$\mathcal{P}_4$: Effectively better.
The BCR must not be ambiguous in the sense that it is not both a best choice as well as a worst choice recommendation.

$\mathcal{P}_5$: Maximally significant.
The BCR is, of all potential best choice recommendation, the most significantly supported one by the marginal "*at least as good as*" relations.

Qualification of a BCR in $\widetilde{G}(X, r(\succsim))$

Let Y be a non empty subset of X, called a choice in $\widetilde{G}$.

- Y is called outranking (resp. outranked) iff for all non retained alternative x there exists an alternative y retained such that $r(y \succsim x) > 0.0$ (resp. $r(x \succsim y) > 0.0$).
- Y is called independent iff for all $x \neq y$ in Y, we observe $r(x \succsim y) \leqslant 0.0$.
- Y is an outranking kernel (resp. outranked kernel) iff Y is an outranking (resp. outranked) and independent choice.
- Y is an outranking (resp. outranked) hyper-kernel iff Y is an outranking (resp. outranked) choice containing chordless circuits of odd order $p \geqslant 1$.

Translating the pragmatic principles in terms of choice qualification

$\mathcal{P}_1$: Elimination for well motivated reasons.
The BCR is an outranking choice.

$\mathcal{P}_{2+3}$: Minimal and stable recommendation.
The BCR is a hyper-kernel.

$\mathcal{P}_4$: Effectivity.
The BCR is a choice which is strictly more outranking than outranked.

$\mathcal{P}_5$: Maximal significance.
The BCR is the most determined one in the set of potential outranking hyper-kernels observed in a given bipolar outraking digraph $\widetilde{G}(X, r(\succsim))$.

Property

Every bipolar strict outranking digraph $\widetilde{G}(X, r(\succsim))$ admits at least one outranking and one outranked hyper-kernel.

The Rubis best choice recommendation (RBCR)

- A strictly outranking hyper-kernel of maximal significance, if it exists, renders a RBCR.
- A RBCR verifies the five pragmatic principles.
- A RBCR is a recommended subset of alternatives which contains the best alternative, provided that it exists.
- A RBCR must not be confused with the actual best choice retained by the decision maker.
- Being only a recommendation, the Rubis best choice approach is only convenient in a progressive decision aiding process.

Computing ranking recommendations

A second traditional solving of a decision problem consists in ranking the alternatives. For this purpose we rely on related work by Lamboray who revisited the prudent orders introduced initially by:

[1] K. Arrow and H. Raynaud, *Social choice and Multicriterion decision-making.* MIT Press (1986).

[2] C. Lamboray, *A comparison between the prudent order and the ranking obtained with Borda's, Copeland's, Slater's and Kemeny's rules.* Mathematical Social Sciences 54 (2007) 1-16.

[3] C. Lamboray, *A prudent characterization of the Ranked Pairs Rule.* Social Choice and Welfare 32 (2009) 129-155.

[4] L.C. Dias and C. Lamboray, *Extensions of the prudence principle to exploit a valued outranking relation.* European Journal of Operational Research Volume 201 Number 3 (2010) 828-837.

Definition and properties of ranking rules

Definition

- A ranking rule is a procedure which aggregates the n marginal at least as good as relations $\geqslant_i$ into a global ranking – either a linear ordering, or a preordering with ties – which "best" exploits the preferential information contained in the bipolar outranking digraph $\widetilde{G}(X, r(\succsim))$.
- A ranking rule is called Condorcet-consistent (Lamboray 2007,2009) if it holds that:
  When the Condorcet graph $G(X, \succsim)$ models a linear order on X, then this linear order is the unique ranking solution resulting from applying the rule.

Condorcet-consistent ranking rules (1)

Kohler's Rule: Optimistic sequential maximin rule.
In: $\widetilde{G}(X, r(\succsim))$; *out*: linear ordering.
At step k (where k goes from 1 to n):

1. Compute for each alternative x the smallest $r(x \succsim y)$ $(x \neq y)$;
2. Select the alternative for which this minimum is maximal. If there are ties select one of these alternatives at random;
3. Put the selected alternative at rank k in the final ranking;
4. Delete the row and the column corresponding to the selected alternative and restart from (1).

Condorcet-consistent ranking rules (2)

Arrow & Raynaud's Rule: Pessimistic sequential minmax rule.
In: $\widetilde{G}(X, r(\succsim))$; *out*: linear ordering.
At step k (where k goes from 1 to n):

1. Compute for each alternative x the largest $r(x \succsim y)$ $(x \neq y)$;
2. Select the alternative for which this maximum is minimal. If there are ties select one of these alternatives at random;
3. Put the selected alternative at rank $n - k + 1$ in the final ranking;
4. Delete the row and the column corresponding to the selected alternative and restart from (1).

Property

Arrow & Raynaud's rule applied to the codual $\widetilde{G}(X, r(\not\precsim))$ is equivalent to Kohler's rule applied to $\widetilde{G}(X, r(\succsim))$ and vice versa.

Condorcet-consistent ranking rules (3)

Tideman's *Ranked Pairs* Rule
In: $\widetilde{G}(X, r(\succsim))$; *out*: linear ordering.

1. Rank in decreasing order the ordered pairs (x, y) of alternatives according to their weighted majority margin $r(x \succsim y)$.
2. Take the linear order compatible with this weak order where ties are resolved by alphabet order of the alternatives.
3. Consider the pairs (x, y) in that order and do the following:
   3.1 If the considered pair creates a cycle with the already blocked pairs, skip this pair;
   3.2 If the considered pair does not create a cycle with the already blocked pairs, block this pair.

Property

Tideman's ranked pairs rule on $\widetilde{G}(X, r(\succsim))$ is equivalent to Dias & Lamboray's Leximin prudent ranking rule (see [4]) applied on $\widetilde{G}(X, r(\not\precsim))$, and vice versa.

Ranking-by-scoring rule

Borda's Rule
In: rank analysis table Q_{xk}; *out*: weak ordering.

- The rank analysis table Q_{xk} gathers the number of times each alternative x is placed at rank k in the linear ordered preferences at hand.
- The Borda score b_x is computed as follows:

$$\forall x \in X, \quad b_x = \sum_{k=1}^{n} Q_{xk} \cdot k.$$

- The Borda ranking $\succeq_B$ is the weak order defined as follows:

$$\forall x, y \in X, \quad (x, y) \in \succeq_B \quad \Leftrightarrow \quad b_x \leqslant b_y.$$

!!! *Borda's ranking-by-scoring rule is not Condorcet-consistent*

Condorcet-consistent ranking-by-scoring rule

Copeland's Rule

In: $G(X, \succsim)$; *out*: weak ordering.

- The idea is that the more a given alternative beats other alternatives at majority the better it should be ranked.
- Similarly, the more other alternatives beat a given alternative at majority, the lower this alternative should be ranked.
- The Copeland score c_x of alternative $x \in X$ is defined as follows:

$$c_x = \#\{y \neq x \in X : r(x \succsim y) > 0\} - \#\{y \neq x \in X : r(y \succsim x) > 0\}$$

- The Copeland ranking $\succeq_C$ is the weak order defined as follows: $\forall x, y \in X, \quad (x, y) \in \succeq_C \quad \Leftrightarrow \quad c_x \geqslant c_y$.

Optimal Condorcet-consistent ranking rules (1)

Kemeny's Rule

In: $\widetilde{G}(X, r(\succsim))$; *out*: linear ordering.

- The idea is finding a compromise ranking O that minimizes the sum of distances to the n marginal outrankings, according to the symmetric difference measure δ (If R_1 and R_2 are two relations, $\delta(R_1, R_2) = |R_1 \oplus R_2| / 2$).
- The Kemeny (also called *median*) order O^* is a solution of the following optimization problem:

$$\begin{array}{rl} \text{maxarg}_O & \sum_{(x,y) \in O} r(x \succsim y) \\ \text{such that} & O \text{ is a linear order on X} \end{array}$$

- The Kemeny order is invariant under the coduality principle.
- Finding a Kemeny order O^* is an NP-complete problem.

Optimal Condorcet-consistent ranking rules (2)

Slater's Rule

In: $G(X, \succsim)$; *out*: linear ordering.

- The idea is to select a ranking that is closest, according to the symmetric difference distance δ, to the Condorcet (median cut) outranking relation $\succsim \equiv \{(x, y) : r(\succsim) > 0.0\}$.
- The Slater order O^* is a solution of the following optimization problem:

$$\begin{array}{rl} \text{minarg}_O & \delta(O, \succsim) \\ \text{such that} & O \text{ is a linear order on X} \end{array}$$

- The distance $\delta(O^*, \succsim)$ is called the Slater index of the outranking relation.
- Computing the Slater index of a relation is an NP-hard problem.

Ranking the potential office sites

Rule	Linear Order	Rank Corr. with $r(\succsim)$
Kohler:	$[G, C, D, B, E, A, F]$	0.96
Arrow & Raynaud:	$[G, C, D, E, B, A, F]$	0.96
Ranked Pairs:	$[C, G, D, B, A, E, F]$	0.97
Leximin:	$[C, G, D, B, A, E, F]$	0.97
Kemeny:	$[C, G, D, B, A, E, F]$	0.97
Copeland:	$[G, C, D, E, A, B, F]$	0.93
Slater:	$[C, G, D, B, A, E, F]$	0.97

Comment

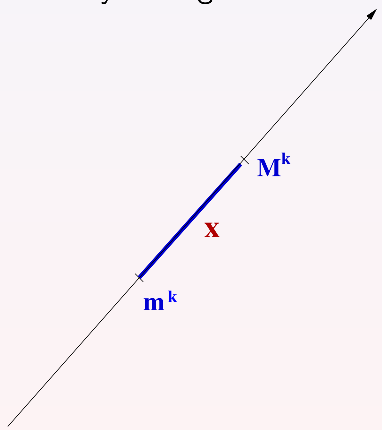
In our example problem, the Ranked Pairs rule delivers an optimal Kemeny order, that is a ranking whose rank correlation with the given bipolarly valued outranking relation is maximal (0.97).

Computing sorting recommendation

- The sorting problematique consists in comparing each alternative to predefined norms of absolute performances;
- These norms are modelled by k ordered categories from the *best* to the *least desired* alternatives;
- A frequently used set of categories consists for instance in: *excellent*, *very good*, *good*, *fair*, *weak*, *very weak*;
- These norms take usually two distinct forms: either, *limiting performances profiles*, or, *central prototypical profiles*;
- We tackle only the case of limiting profiles here, the other case being more related to a *supervised ordered clustering problematique*.

K-Sorting on a single criterion

Category K is an interval $[m^k; M^k[$ on the criterion performance scale; x is a measured performance.
We may distinguish three sorting situations:



1. $x < m^k$ (and $x < M^k$)
   The performance x is lower than category K;
2. $x \geqslant m^k$ and $x < M^k$
   The performance x belongs to category K;
3. ($x \geqslant m^k$ and) $x \geqslant M^k$
   The performance x is higher than category K.

If the relation $<$ is the dual of $\geqslant$, it will be sufficient to check that $x \geqslant m_k$ as well as $x \not\geqslant M_k$ are true for x to be a member of K.

K-Sorting on p criteria

Let $m^k = (m^k_1, m^k_2, ..., m^k_p)$ denote the lower limits and $M^k = (M^k_1, M^k_2, ..., M^k_p)$ the corresponding upper limits of category K on the criteria.
Let $\widetilde{G}(X \cup m^k \cup M^k, r(\succsim))$ represent the bipartite digraph giving the bipolarly-valued outrankings between X and the lower and upper category limits.

Property

That alternative $x \in X$ belongs to category K may be characterised as follows:

$$r(x \in K) \; = \; \min\big(r(x \succsim m^k), r(x \not\precsim M^k)\big)$$

Follows, indeed, from the coduality principle:
$r(x \not\precsim M^k) = r(M^k \succnsim x)$.

Multiple criteria K-Sorting algorithm

1. **Input**: a set X of n alternatives with a performance table on a family of p criteria and a set $\mathcal{C}$ of k empty categories K with lower limit m^k and upper limit M^k.
2. **For each** alternative $x \in X$ **and each** category $K \in \mathcal{C}$
   2.1 $r(x \in K) \quad \leftarrow \quad \min\big(r(x \succsim m^k), r(x \not\precsim M^k)\big)$
   2.2 if $r(x \in K) \; \geqslant \; 0$:
   add x to category K
3. **Output**: $\mathcal{C}$

Comment

1. *The complexity of the K-Sorting algorithm is linear: $\mathcal{O}(nkp)$.*
2. *In case, $\mathcal{C}$ represents p partitions of the criteria measurement scales, i.e. the upper limits of the preceding category correspond to the lower limits of the succeeding ones, there is a potential for reducing the complexity even more.*

Revisiting the best office choice problem

Supervised 6-sorting of the office sites in descending order using a common partition of six lower closed %-classes of equally spaced performance limits on each objective's scale.

Limits	Category
]> - 100%]	[]
]100% - 80%]	[*G*]
]80% - 60%]	[*B*, *C*, *D*, *E*, *F*, *G*]
]60% - 40%]	[*A*, *B*, *D*, *E*, *F*]
]40% - 20%]	[*A*, *E*, *F*]
]20% - 0%]	[]

Comment

1. *Site G is 'best' sorted between* 60% *and* 100%, *followed by site C sorted between* 60% *and* 80%.
2. *Then come sites B and D sorted between* 40% *and* 80%, *followed by sites E and F sorted between* 20% *and* 80%.
3. *Finally, site A is sorted between* 20% *and* 60%.

Properties of K-Sorting result

1. *Coherence*: Each alternative is always sorted into a possibly empty subset of adjacent categories.
2. *Weak Unicity*: In case of non overlapping categories and the absence of indeterminate bipolar outrankings, i.e. $r \neq 0$, every alternative is sorted into at most one category;
3. *Unicity*: If the categories represent a discriminated partition of the performance measurement scales on each criterion and $r \neq 0$, then every alternative is sorted into exactly one category;
4. *Independance*: The sorting result for alternative x, is independent of the other alternatives' sorting results.
5. *Monotonicity*: If $r(x \geqslant y) = 1$, then alternative x is sorted into a category which is at least as high ranked as the category into which is sorted alternative y.
6. *Stability*: If a category is dropped from $\mathcal{C}$, the content of the remaining categories will not change thereafter.

Conclusions

- Similarly to MAVT, the outranking approach stresses the necessicity to follow a consistent and systematic approach for evaluating the performances of the potential decision alternatives.
- Similarly to MAVT, the outranking approach allows to model costs and benefits with the help of multiple qualitative and/or quantitative performance criteria.
- Contrary to MAVT, the outranking approach does not make the assumption that the evaluations on all the criteria must be commensurable in order to model global preferences.
- Contrary to weighted scoring approaches, the significance of the criteria (not to be confused with substitution rates) in the global outranking credibility calculus does not need to take into account type and scope of the marginal performance measurement scales.

Conclusions

- By adopting a pairwise comparison approach à la Condorcet, we abandon the idea of complete comparability and transitivity of the preferences and receive in return the independence of all pairwise preferential statements from irrelevant alternatives (see Arrows impossibility theorem).
- Taking into account performance discrimination thresholds allows to efficiently model imprecision, uncertainties and even very large positive and negative differences in the performance data.
- The bipolar characteristic valuation $r(\succsim)$ in $[-1.0; +1.0]$ allows with the median value 0.0 to handle safely highly contradictory as well as missing data.
- Due the verification of the coduality principle, the bipolar outranking relation allows us to solve efficiently multiple criteria based choice, ranking, sorting and also clustering problems.

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- A random generator for XMCDA-2.0 encoded performance tableaux is available on the following page: `http://leopold-loewenheim.uni.lu/rubisServer/random/randomPerfTabGeneratorAjax.html`