On weakly ordering with multiple criteria quantiles sorting

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Abstract. We apply order statistics for sorting a set X of n potential deicison actions, evaluated on p incommensurable performance criteria, into k quantile equivalence classes, based on pairwise outranking characteristics involving the quantile class limits observed on each criterion. Thus we may implement a weak ordering algorithm of complexity $\mathcal{O}(npk)$.

keywords*:* multiple criteria decision aid; multiple criteria weakly ordering; quantiles sorting; bipolar-valued outranking.

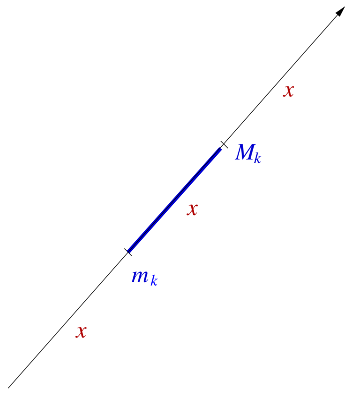
1 K-Sorting on a single criterion

1.1 Sorting into a single category

A single criterion sorting category K is a (usually) lower-closed interval $[m_k; M_k[$ on a real-valued measurement scale. If x is a measured performance on this scale, we may distinguish three sorting situations:

1. $x < m_k$ (and $x < M_k$): The performance x is lower than category K;
2. $x \geqslant m_k$ and $x < M_k$: The performance x belongs to category K;
3. $x \geqslant M_k$ (and $x \geqslant m_k$) : The performance x is higher than category K.

As the relation $<$ is the *dual* of $\geqslant$, it will be sufficient to check that $x \geqslant m_k$ as well as $x \ngeqslant M_k$ are true for x to be considered a member of category K.



Upper-closed categories, like in old fashioned official statistics, may also be considered. In this case it is sufficient to check that $m_k \ngeqslant x$ as well as $M_k \geqslant x$ are true for x to be considered a member of category K. It is worthwhile noticing that a category K such that $m_k = M_k$ is hence always empty by definition.

In order to be able to properly sort over the complete range of values to be sorted, we will need to use a special, two-sided closed last, respectively first, category.

1.2 Sorting into quantile categories

Let $\mathcal{K} = \{K_1, ..., K_c\}$ be a non trivial partition of the criterion's performance measurement scale into $c \geq 2$ ordered categories K_k – i.e. lower-closed intervals $[m_k; M_k[$ – such that $m_k < M_k$, $M_k = m_{k+1}$ for $k = 0, ..., c-1$ and $M_c = +\infty$. And, let $A = \{a_1, a_2, a_3, ...\}$ be a finite set of not all equal performance measures observed on the scale in question.

Property 1. For all performance measure $x \in A$ there exists now a unique k such that $x \in K_k$. If we assimilate, like in descriptive statistics, all the measures gathered in a category K_k to the central value of the category – i.e. $(m_k + M_k)/2$ – the sorting result will hence define a weak order (complete preorder) on A.

Let $\mathcal{Q} = \{Q_0, Q_1, ..., Q_c\}$ denote the set of $c+1$ increasing order-statistical quantiles (percentiles) – like quartiles or deciles – we may compute from the ordered set A of performance measures observed on a performance scale. If $Q_0 = \min(X)$ we may, with the following intervals: $[Q_0; Q_1[$, $[Q_1; Q_2[$, ..., $[Q_{c-1}; +\infty[$, hence define a set of c of lower-clased sorting categories. And, in the case of upper-closed categories, if $Q_c = \max(X)$, we would obtain the intervals $]-\infty; Q_1]$, $]Q_1; Q_2]$, ..., $]Q_{c-1}; Q_c]$. The corresponding sorting of A will result, in both cases, in a repartition of all x measures into the c quantile categories K_k for $k = 1, ..., c$.

Example 1. Let $A = \{\ a_7 = 7.03,\ a_{15} = 9.45,\ a_{11} = 20.35,\ a_{16} = 25.94,\ a_{10} = 31.44,\ a_9 = 34.48,\ a_{12} = 34.50,\ a_{13} = 35.61,\ a_{14} = 36.54,\ a_{19} = 42.83,\ a_5 = 50.04,\ a_2 = 59.85,\ a_{17} = 61.35,\ a_{18} = 61.61,\ a_3 = 76.91,\ a_6 = 91.39,\ a_1 = 91.79,\ a_4 = 96.52,\ a_8 = 96.56,\ a_{20} = 98.42\}$ be a set of 20 performance measures observed on a given criterion. The lower-closed category limits we would obtain with quartiles ($c = 4$) are: $Q_0 = 7.03 = a_7$, $Q_1 = 34.485$, $Q_2 = 54.945$ (median performance), and $Q_3 = 91.69$. And the sorting into these four categories defines on A a complete preorder with the following four equivalence classes: $K_1 = \{a_7, a_{10}, a_{11}, a_{10}, a_{15}, a_{16}\}$, $K_2 = \{a_5, a_9, a_{13}, a_{14}, a_{19}\}$, $K_3 = \{a_2, a_3, a_6, a_{17}, a_{18}\}$, and $K_4 = \{a_1, a_4, a_8, a_{20}\}$.

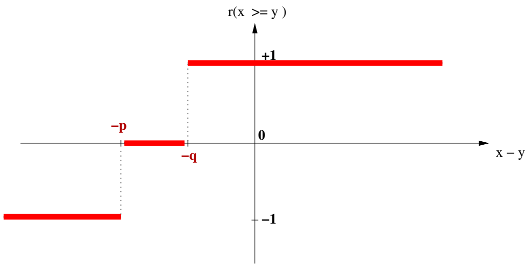
1.3 Sorting with imprecise and uncertain performance measures

Uncertainties, imprecision as well as measurement errors and inaccuracy may often affect the set of given performance measures to sort. To take these into account we are going to introduce performance discrimination thresholds.

Let x and y be two performance measurement with respect to a given criterion. Let $0 \leq q < p \leq M_c$ represent the indifference (q), respectively the preference (p), discrimination threshold observed when measuring with out loss of genericity performances on an increasing real-valued scale in the range 0 to M_c. Both, these performance discrimination thresholds characterise a homogeneous double threshold ordering $\succeq$ on X in the following way:

$$r(x \succeq y) = \begin{cases} +1 & \text{if} \quad x + q \geq y \\ -1 & \text{if} \quad x + p \leq y \\ 0 & \text{otherwise.} \end{cases} \qquad (1)$$

- $+1$ signifies that "x is *performing at least as good as* y",
- -1 signifies that "x is *not performing at least as good as* y",
- 0 signifies that "it is *unclear* whether, on criterion i, x is performing at least as good as y".



To assess now the sorting situation of a performance measure x with respect to a sorting category K_k defined by the interval $[m_k; M_k[$, we use a bipolar characteristic function r that is defined as follows for all $x \in X$ and $K_k \in \mathcal{K}$:

$$r(x \in K_k) = r\big((x \succeq m_k) \ \wedge \ (x \not\succeq M_k) \big) \tag{2}$$
$$= \min \big(r(x \succeq m_k), r(x \not\succeq M_k) \big) \tag{3}$$
$$= \min \big(r(x \succeq m_k), -r(x \succeq M_k) \big). \tag{4}$$

We hence get $r(x \in K_k) = +1$ if and only if $r(x \succeq m_k) = +1$ and $r(x \not\succeq M_k) = +1$, i.e. x appears within the limits defining category K_k. If $r(x \in K_k)$ or $r(x \succeq m_k)$ equals -1, then x is certainly situated outside the limits defining category K_k. With upper-closed categories, we would use the rule: $r(x \in K_k) = \min \big(-r(m_k \succeq x), r(M_k \succeq x) \big)$.

It may happen that $r(x \in K_k) = 0$. This occurs when $r(x \succeq m_k)$ or $r(x \not\succeq M_k)$ equals 0. In this case the sorting result for performance x is indeterminate with respect to category K_k.

Example 2. Let us assume that on the previous set A of 20 performance measures (see *Example* 1) we observe an indifference discrimination threshold of $q = 2.5$, and a preference discrimination threshold of $p = 5.0$. In this case we observe that the difference between a_{10} and Q_1 ($|31.44 - 34.485| = 3.045$) is in fact larger than the indifference threshold (2.5), but also smaller than the preference threshold (5.0). Hence, $r(a_{10} \succeq Q_1) = 0$. As $Q_1 = M_1 = m_2$, we may for sure conclude that a_{10} is performing better than $m_1 = 7.03$ and less performing than $M_1 = 50.04$. However, we are not sure whether a_{10} is in fact *less performing* than or *at least as good performing* as M_1. As a consequence, we both get $r(a_{10} \in K_1) = 0$ and $r(a_{10} \in K_2) = 0$. It is thus uncertain whether a_{10} may be sorted in K_1 or in K_2. A similar situation happens when sorting measure a_5.

Property 2. When $\mathcal{K}$ defines a partition of the performance scale, using Rule 4: $r(x \in K_k) \geq 0$ for sorting a performance x into a category K_k , results in a sorting result where each performance measure x is either sorted into one, or spread indeterminately over two or more adjacent categories.

Example 3. Resorting again the previous 20 performance values into quartiles categories (see *Example* 1), this time with discrimination thresholds $q = 2.5$ and $p = 5.0$, gives on A again four equivalence classes:

$$K_4 = \{a_1, a_4, a_6, a_8, a_{20}\}.$$
$$K_3 = \{a_2, a_3, \mathbf{a_5}, a_{17}, a_{18}\}$$
$$K_2 = \{\mathbf{a_5}, a_9, \mathbf{a_{10}}, a_{12}, a_{13}, a_{14}, a_{19}\}$$
$$K_1 = \{a_7, \mathbf{a_{10}}, a_{11}, a_{15}, a_{16}\}$$

Performance measures a_5 and a_{10} are indeed sorted indeterminately into categores K_1 or K_2, respectively K_2 or K_3. Notice furthermore that measure $a_{12} = 34.48$, being now considered "*at least as good*" as the upper limit $M_1 = 34.485$ of K_1, has consequently been upgraded from K_1 to K_2.

2 Multiple criteria K-sorting

2.1 Overall performance comparison concordance

Let X be a finite set of objects to be sorted and let $F = \{1, ..., n\}$ be a finite and coherent family of n performance criteria. On each criterion i in F, the objects are evaluated on a real performance scale $[0; M^i]$, supporting an indifference threshold q_i and a preference threshold p_i such that $0 \leq q_i < p_i \leq M^i$. The performance of object x on criterion i is denoted x_i. Each criterion i is thus characterising a marginal double threshold ordering $\succeq_i$ on X as defined in *Equation* 1.

Globally performing "at least as good as":
Furthermore, each criterion i in F carries a rational significance w_i such that $0 < w_i < 1.0$ and $\sum_{i \in F} w_i = 1.0$ which he contributes to the characterisation of a global "at least as good as" relation a global $\succeq$ in the following way:

$$r(x \succeq y) \;=\; \sum_{i \in F} \big[\, w_i \cdot r(x_i \succeq_i y_i) \,\big] \tag{5}$$

$r > 0$ signifies x is *globally performing at least as good as* y,
$r < 0$ signifies that x is *not globally performing at least as good as* y,
$r = 0$ signifies that it is *unclear* whether x is globally performing at least as good as y.

Globally performing less than:
Each criterion i is characterising a marginal homogeneous double threshold ordering $\prec_i$ (*less than*) on A in the following way:

$$r(x_i \prec_i y_i) \quad = \quad \begin{cases} +1 & \text{if} \quad x_i + p_i \leq y_i \\ -1 & \text{if} \quad x_i + q_i \geq y_i \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

And, the *global less than* relation ($\prec$) is defined as follows:

$$r(x \prec y) = \sum_{i \in F} \big[w_i \cdot r(x_i \prec_i y_i) \big] \tag{7}$$

Property 3 (Bisdorff 2013). The global "*less than*" relation $\prec$ is the dual ($\not\succeq$) of the global "*at least as good as*" relation $\succeq$.

The property follows readily from the fact that the marginal relation $\prec_i$ is the dual of the marginal $\succeq_i$ relation (see *Equations* (1) and (6)). We also say that the global "at least as good as" relation ($\succeq$) verifies the coduality principle, in the sense that its asymmetric part, the strict "better than" relation ($\succ$) is identical to the converse of the negation of it (**?**).

Let $\mathbf{m}_k = (m_{1,k}, m_{2,k}, ..., m_{n,k})$ denote the *lower limits* and $\mathbf{M}_k = (M_{1,k}, M_{2,k}, ..., M_{n,k})$ the corresponding *upper limits* of category K_k on a family of n criteria.

Property 4. Similar to the single criterion case, that object $x \in X$ belongs to lower-closed category K_k may now, globally, be characterised as follows:

$$r(x \in K_k) \;=\; \min\big(r(x \succeq \mathbf{m}_k), -r(x \succeq \mathbf{M}_k) \big) \tag{8}$$

If K_k is upper-closed the formula becomes:

$$r(x \in K_k) \;=\; \min\big(-r(\mathbf{m}_k \succeq x), r(\mathbf{M}_k \succeq x) \big) \tag{9}$$

2.2 Observing non compensable performance differences

In order to take into account large non-compensable marginal performance differences, we define a single threshold order, denoted $\ll_i$ and which represents a marginal *considerably less performing* situation observed on a criterion i, as follows:

$$r(x \ll_i y) = \begin{cases} +1 & \text{if} \quad x_i + v_i \leq y_i \\ -1 & \text{if} \quad x_i - v_i \geq y_i \\ 0 & \text{otherwise.} \end{cases} \quad . \tag{10}$$

And a corresponding dual *considerably better performing* situation $\gg_i$ characterised as:

$$r(x \gg_i y) = \begin{cases} +1 & \text{if} \quad x_i - v_i \geq y_i \\ -1 & \text{if} \quad x_i + v_i \leq y_i \\ 0 & \text{otherwise.} \end{cases} \quad . \tag{11}$$

Vetoes and counter-vetoes situations:
A global *veto* or *counter-veto* situation is now defined as follows:

$$r(x \ll y) = \bigotimes_{i \in F} r(x \ll_i y) \tag{12}$$
$$r(x \gg y) = \bigotimes_{i \in F} r(x \gg_i y) \tag{13}$$

where $\otimes$ represents the epistemic polarising (?) or symmetric maximum (?) operator:

$$r \otimes r' = \begin{cases} \max(r, r') & \text{if} \quad r \geqslant 0 \wedge r' \geqslant 0, \\ \min(r, r') & \text{if} \quad r \leqslant 0 \wedge r' \leqslant 0, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Let $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ be two multiple criteria performance measures.

1. $r(x \ll y) = 1$ iff there exists a criterion i such that $r(x_i \ll_i y_i) = 1$ and there does not exist otherwise any criteria j such that $r(x_j \gg_j y_j) = 1$.
2. Conversely, $r(x \gg y) = 1$ iff there exists a criterion i such that $r(x_i \gg_i y_i) = 1$ and there does not exist otherwise any criteria j such that $r(x_j \ll_j y_j) = 1$.
3. $r(x \gg y) = 0$ if either we observe no very large performance differences or we observe at the same time, both a very large positive and a very large negative performance difference.

Property 5. $r(\not\ll)^{-1}$ is identical to $r(\gg)$, i.e. relations $\ll$ and $\gg$ verify in fact the coduality principle.

2.3 The bipolar global outranking relation $\succsim$

Let again $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ be performance measures observed on a family F of n performance criteria. From an epistemic point of view, we say that:

1. measure x *outranks* measure y, denoted $(x \succsim y)$, if
   (a) a significant majority of criteria validates a global outranking situation between x and y, and
   (b) no considerable counter-performance is observed on a discordant criterion,
2. object x *does not outrank object* y, denoted $(x \not\succsim y)$, if

(a) a significant majority of criteria invalidates a global outranking situation between x and y, and
(b) no considerably better performing situation is observed on a concordant criterion.

Hence, the bipolar-valued characteristic $r(\succsim)$ is defined as follows:

$$r(x \succsim y) = \begin{cases} 0 \quad \text{if } \big[\exists i \in F : r(x \ll_i y)\big] \wedge \big[\exists j \in F : r(x \gg_j y)\big], \\ \big[\, r(x \succeq y) \circledvee -r(x \ll y) \,\big], \text{otherwise.} \end{cases} \tag{15}$$

And in particular,

$r(x \succsim y) = r(x \succeq y)$ if no very large positive or negative performance differences are observed,
$r(x \succsim y) = 1$ if $r(x \succeq y) \geqslant 0$ and $r(x \gg y) = 1$,
$r(x \succsim y) = -1$ if $r(x \succeq y) \leqslant 0$ and $r(x \ll y) = 1$.

We call *weakly complete* a binary relation R on A when its bipolar characteristic function verifies $r(xRy) < 0) \Rightarrow r(yRx) \geqslant 0$ for all $(x, y) \in A^2$.

Property 6. The biplar outranking relation $\succsim$ defines on a given set A of multiple criteria performance measures a weakly complete binary relation.

The property follwos directly from the facts that: i) the global at least as good relation $\succeq$ is weakly complete, and ii) the polarization with considerable performance difference via Equation (2.3) does not change any sign of the characteristic r-values. The bipolar outranking relation verifies furthermore the coduality principle.

Proposition 1 (Bisdorff 2013). *The dual ($\not\succsim$) of the bipolar outranking relation $\succsim$ is identical to the strict converse outranking $\precnsim$ relation.*

Proof.

$$\begin{aligned} r(x \not\succsim y) = -r(x \succsim y) \;&=\; -\big[r(x \succeq y) \circledvee -r(x \ll y)\big] \\ &= \big[-r(x \succeq y) \circledvee r(x \ll y)\big] \\ &= \big[r(x \not\succeq y) \circledvee -r(x \gg y)\big] \\ &= \big[r(x \prec y) \circledvee r(x \not\gg y)\big] \;=\; r(x \precnsim y). \end{aligned}$$

Corollary 1. *The bipolar characteristic of x belonging to a lower-closed sorting category K_k may be assessed :*

$$r(x \in K_k) = \min\big(r(x \succsim \mathbf{m}_k), r(x \not\succsim \mathbf{M}_k) \big), \tag{16}$$

repectively,

$$r(x \in K_k) = \min\big(r(\mathbf{m}_k \not\succsim x), r(\mathbf{M}_k \succsim x) \big) \tag{17}$$

in the case of upper-closed sorting categories.

Example 4. Let us consider a set $A = \{a_1, a_2, a_3, a_4, a_5\}$ of decision actions randomly evaluated on a coherent set $F = \{1, 2, 3\} of equisignificant criteria such that criteria 1$ and 2 support an ordinal performance measurement scale coded respectively as $0, 10, 20, ..., 100$. The preference dicrimination threshold is supposed to be 1 and there is no indifference or veto threshold observed on these criteria. Criterion 3 is, however, of cardinal type with a rational performance measurement scale between 0.0 and 100.0 supporting an indifference discrimination threshold of 5.53, a preference discrimination threshold of 8.93 and a veto threshold of 61.94. These discrimination thresholds are chosen so as to touch each one 10% of all performance differences observed on this criterion.

Random Performance Tableau

$F \times A$	a_1	a_2	a_3	a_4	a_5
1	60	60	30	40	80
2	80	50	20	70	90
3	65.82	24.89	27.02	9.21	75.51

Quintile performance limits per criterion

Quintiles	criterion 1	criterion 2	criterion 3
$]<;0.20]$	$]<;44]$	$]<;54]$	$]<;25.32]$
$]0.20;0.40]$	$]44;60]$	$]54;74]$	$]25.32;42.54]$
$]0.40;0.60]$	$]60;72]$	$]74;86]$	$]42.54;71.63]$
$]0.60;0.80]$	$]72;80]$	$]86;90]$	$]71.63;75.51]$
$]0.80;1.00]$	$]80;80]$	$]90;90]$	$]75.51;75.51]$

Let us try to sort action a_1's performance measures into upper-closed quintiles. Following criterion 1, a_1 belongs to quintile $]0.20;0.40]$, whereas on criteria 2 and 3, it belongs to quintile $]0.40;0.60]$. If we compute the sorting characteristic function values for all the quintiles, we get the following results:

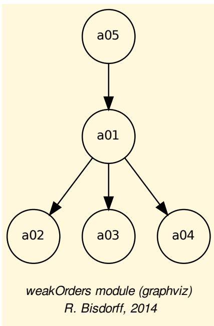
$x \in A$	$]m_k;M_k[$	$r(x \succsim m_k)$	$r(x \not\succsim M_k)$	$r(x \in K_k)$
a_1	$]<;0.20]$	$+\mathbf{1.0}$	-1.0	-1.0
	$]0.20;0.40]$	$+\mathbf{1.0}$	-0.33	-0.33
	$]\mathbf{0.40};\mathbf{0.60}]$	$+\mathbf{0.33}$	$+\mathbf{1.0}$	$+\mathbf{0.33}$
	$]0.60;0.80]$	-1.0	$+\mathbf{1.0}$	-1.0
	$]0.80;1.00]$	-1.0	$+\mathbf{1.0}$	-1.0

And, indeed, until quintile $]0.40;0.60]$, action a_1's performances positively *outrank* the lower limits of the preceeding quintiles. Similarly, from quintile $]0.40;0.60]$ on, action a_1's performances positively *do not outrank* the upper limits of the succeeding quintiles. Hence, quintile $]0.40;0.60]$ is the only class into which action a_1 may be positively sorted: $r(a_1 \succsim 0.40) = +0.33$ and $r(a_1 \not\succsim 0.60) = +1.0$. Hence, $r(a_1 \in]0.40;0.60]) = \min(+0.33, +1.0) = +0.33$.

The complete sorting result for all the five actions in A is shown below:

Sorting results in descending order:

Quintile	Sorted Actions
$]0.80;1.00]$	-
$]0.60;0.80]$	a_5
$]0.40;0.60]$	a_1
$]0.20;0.40]$	-
$]<;0.20]$	$a_2,\ a_3,\ a_4$



3 Multiple criteria quantiles sorting

3.1 The multicriteria K-Sorting algorithm

1. **Input**: a set A of n decision actions evaluated on a family of p criteria and a set $\mathcal{K}$ of k empty lower-closed categories K_k with lower and upper limits m_k and M_k.
2. **For each** action $x \in A$ **and each** category $K_k \in \mathcal{K}$
   (a) $r(x \in K_k) \quad \leftarrow \quad \min\big(r(x \succsim m_k), r(x \not\succsim M_k)\big)$

(b) if $r(x \in K_k) \geqslant 0$:
add x to category K

3. **Output**: $\mathcal{K}$

1. The complexity of the K-Sorting algorithm is linear: $\mathcal{O}(nkp)$.
2. In case, $\mathcal{K}$ represents p partitions of the criteria measurement scales, i.e. the upper limits of the preceding category correspond to the lower limits of the succeeding ones, there is a potential for reducing the complexity even more.

3.2 Properties of K-Sorting result

1. *Coherence*: Each action is always sorted into a possibly empty subset of adjacent categories.
2. *Weak Unicity*: In case of non overlapping categories and the absence of indeterminate bipolar outrankings, i.e. $r \neq 0$, every action is sorted into at most one category;
3. *Unicity*: If the categories represent a discriminated partition of the measurement scales on each criterion and $r \neq 0$, then every action is sorted into exactly one category;
4. *Independance*: The sorting result for action x, is independent of the other actions' sorting results.
5. *Monotonicity*: If $r(x \succsim y) = 1$, then action x is sorted into a category which is at least as high ranked as the category into which is sorted action y.
6. *Stability*: If a category is dropped from $\mathcal{K}$, the contents of the remaining categories will not change thereafter.

Example 5. We consider again a set A of performance measures taken with respect to three equi-significant criteria aupporting the discrimination thresholds shown in the table below. If we sort these measures into twentiles, we obtain following results:

Measure	crit. 1	crit. 2	crit. 3
pref.	3.14	3.27	4.46
ind.	1.42	1.45	1.88
veto	54.35	56.61	53.35
a_1	-46.64	21.60	65.87
a_2	-82.98	32.00	43.99
a_3	-60.33	56.38	71.92
a_4	-79.58	57.36	66.46
a_5	-32.92	14.59	71.84
a_6	-43.71	30.29	76.31
a_7	-83.18	27.88	35.07
a_8	-29.18	50.01	26.08
a_9	-73.25	85.37	79.76
a_{10}	-48.97	81.57	79.95
a_{11}	-22.82	83.95	42.69
a_{12}	-28.30	30.48	61.18
a_{13}	-45.28	58.32	56.17
a_{14}	-45.55	52.32	43.06
a_{15}	-48.96	72.86	87.40
a_{16}	-65.87	62.77	18.98
a_{17}	-57.16	26.90	25.81
a_{18}	-29.76	30.57	69.81
a_{19}	-84.56	62.07	50.83
a_{20}	-29.92	61.66	70.16

Twentiles sorting results:

```
]0.95 - 1.00] []
]0.90 - 0.95] []
]0.85 - 0.90] []
]0.80 - 0.85] []
]0.75 - 0.80] ['a09', 'a10', 'a15']
]0.70 - 0.75] ['a15']
]0.65 - 0.70] []

]0.60 - 0.65] ['a16', 'a19']
]0.55 - 0.60] ['a03', 'a16', 'a19']
]0.50 - 0.55] ['a03', 'a20']
]0.45 - 0.50] ['a04', 'a20']
]0.40 - 0.45] []
]0.35 - 0.40] ['a13']
]0.30 - 0.35] ['a01', 'a13']
]0.25 - 0.30] ['a01', 'a02', 'a06', 'a14']
]0.20 - 0.25] ['a02', 'a05']
]0.15 - 0.20] ['a02', 'a05', 'a11', 'a12', 'a18']
]0.10 - 0.15] ['a05', 'a07', 'a12', 'a18']
]0.05 - 0.10] ['a17']
]< - 0.05]    ['a08']
```