On weakly ordering by choosing from valued pairwise outranking situations

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Abstract

Polarised outrankings with considerable performance differences (weighted majority margins with vetoes and counter-vetoes) appear as weakly complete (and reflexive) relations ([7]). For any two potential decision actions x and y, if x does effectively not outrank y then it is not the case that y does effecively not outrank x. Constructing weak orderings (rankings with potential ties) from such outranking relations consists in computing, hence, a transitive closure of the given outranking relation. Determining optimal transitive closures is a computational difficult problem ([6]). However, global scoring methods based on average ranks (like Borda scores) or average netflows like in the PROMETHEE approach, may easily deliver such a heuristic closure. Now, such weak orderings may also result from the iterated application of a certain choice procedure, an approach called *ranking-by-choosing* ([1]). In this contribution we shall present such a new ranking-by-choosing approach based on the *Rubis* best choice method ([3]).

Contents

1 Illustration

1.1 Sample outranking relation

Let $X = \{a_1, ..., a_7\}$ be seven potential decision actions evaluated on three cost criteria (g_1, g_4, g_5) of equi-significance 1/6 and two benefit criteria (g_2, g_3) of equi-signifiance 1/4. The given performance tableau is shown in Table 1 below.

Table 1: Sample performance tableau

Objectives	**Costs**			**Benefits**	
Criteria	$g_1(\downarrow)$	$g_4(\downarrow)$	$g_5(\downarrow)$	$g_2(\uparrow)$	$g_3(\uparrow)$
weights×12	*2.0*	*2.0*	*2.0*	*3.0*	*3.0*
indifference	3.41	4.91	-	-	2.32
preference	6.31	8.31	-	-	5.06
veto	60.17	67.75	-	-	48.24
a_1	22.49	36.84	7	*8*	43.44
a_2	*16.18*	19.21	*2*	8	*19.35*
a_3	29.41	54.43	3	4	33.37
a_4	*82.66*	*86.96*	*8*	6	48.50
a_5	47.77	82.27	7	7	*81.61*
a_6	32.50	*16.56*	6	*8*	34.06
a_7	35.91	27.52	*2*	*1*	50.82

The resulting bipolar outranking relation S is shown below. We may notice in Table 2 that:

1. a_6 is a *Condorcet winner*,

Table 2: r-valued bipolar outranking relation

$r(\mathrm{S}) \times 12$	a_1	a_2	a_3	a_4	a_5	a_6	a_7
a_1	$-$	***0***	$+8$	$+12$	$+6$	$+4$	-2
a_2	$+6$	$-$	$+6$	$+12$	**0**	$+6$	$+6$
a_3	-8	-6	$-$	**0**	-12	$+2$	-2
a_4	-12	-12	**0**	$-$	-8	-12	**0**
a_5	-2	**0**	$+12$	$+12$	$-$	-6	**0**
a_6	$+2$	$+4$	$+8$	$+12$	$+6$	$-$	$+2$
a_7	$+2$	-2	$+2$	$+6$	**0**	$+2$	$-$

2. a_2 is a *weak* Condorcet winner,

3. a_4 is a *weak* Condorcet looser.

1.2 Ranking-by-choosing

Let X_1 be the set X of potential decision actions we wish to rank.

While the remaining set X_i ($i = 1, 2, ...$) of decision actions to be ranked is not empty, we extract from X_i the best (B_i), respectively worst (W_i) RUBIS choice recommendations and set $X_{i+1} = X_i - B_i$, respectively $X_{i+1} = X_i - W_i$.

Both iterations determine, hence, two – usually slightly different – opposite weak orderings on X:

1. a ranking-y-best-choosing order and,

2. a ranking-by-worst-rejecting order.

Fusion of best and worst choice rankings

Ranking by *recursively choosing*:

```
1st Best Choice ['a02','a05']
 2nd Best Choice ['a06']
   3rd Best Choice ['a07']
     4th Best Choice ['a01']
       5th Best Choice ['a03','a04']
```

Ranking by *recursively rejecting*:

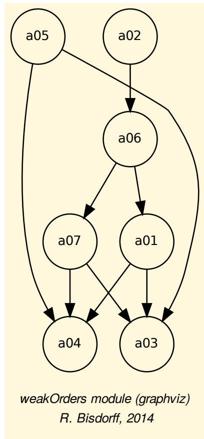
```
Last Choice ['a03','a04']
 2nd Last Choice ['a05','a07']
   3rd Last Choice ['a01']
     4th Last Choice ['a06']
       5th Last Choice ['a02']
```

We may fuse both rankings, the first and the converse of the second, with the help of the *epistemic conjunction* operator ($\owedge$)[1] to make apparent a valued relation R which represents a *weakly complete and transitive closure* of the given bipolar valued outranking.

1.3 Partial ranking result

Table 1.3: Weakly complete transitive closure of S

r(R)	a_2	a_5	a_6	a_1	a_7	a_3	a_4
a_2	$-$	**0**	+6	+6	+6	+6	+12
a_5	**0**	$-$	**0**	**0**	**0**	+12	+12
a_6	-4	**0**	$-$	+2	+2	+8	+12
a_1	0	**0**	-4	$-$	**0**	+8	+12
a_7	-2	**0**	-2	**0**	$-$	+2	+6
a_3	-6	-12	-2	-8	-2	$-$	**0**
a_4	-12	-8	-12	-12	0	**0**	$-$



Notice the contrasted ranks of *action* a_5 (first best as well as second last), indicating a lack of comparability, which becomes apparent in the conjunctive epistemic fusion R of both weak orderings shown in the Table 1.3 above and illustrated in the corresponding Hasse diagram.

2 The setting

2.1 Weakly complete relations

Let $X = \{x, y, z, ...\}$ be a finite set of m decision alternatives.

We define a binary relation R on X with the help of a bipolar characteristic function r taking values in the rational interval $[-1.0; 1.0]$.

Bipolar semantics: For any pair $(x, y) \in X^2$,

[1]Let ϕ and ψ be two logical formulas:

$$\phi \owedge \psi = \begin{cases} (\phi \wedge \psi) & \text{if} \quad (\phi \wedge \psi) \text{ is true;} \\ (\phi \vee \psi) & \text{if} \quad (\neg\phi \wedge \neg\psi) \text{ is true;} \\ \text{Indeterminate} & \text{otherwise.} \end{cases}$$

1. $r(x \,\mathrm{R}\, y) \;=\; +1.0$ means $x \,\mathrm{R}\, y$ valid for sure,
2. $r(x \,\mathrm{R}\, y) \;>\; 0.0$ means $x \,\mathrm{R}\, y$ more or less valid,
3. $r(x \,\mathrm{R}\, y) \;=\; 0.0$ means both $x \,\mathrm{R}\, y$ and $x \not\mathrm{R}\, y$ indeterminate,
4. $r(x \,\mathrm{R}\, y) \;<\; 0.0$ means $x \not\mathrm{R}\, y$ more or less valid,
5. $r(x \,\mathrm{R}\, y) \;=\; -1.0$ means $x \not\mathrm{R}\, y$ valid for sure.

Boolean operations: Let ϕ and ψ be two relational propositions.

1. $r(\neg\phi) \;=\; -r(\phi)$.
2. $r(\phi \vee \psi) \;=\; \max\big(r(\phi), r(\psi)\big)$,
3. $r(\phi \wedge \psi) \;=\; \min\big(r(\phi), r(\psi)\big)$.

Let R be an r-valued binary relation defined on X.

Definition 1. We say that R is *weakly complete* on X if, for all $(x, y) \in X^2$, **either** $r(x \,\mathrm{R}\, y) \geqslant 0.0$ **or** $r(y \,\mathrm{R}\, x) \geqslant 0.0$.

Examples:

1. Marginal semi-orders observed on each criterion,
2. Weighted condordance relations,
3. Polarised outranking relations,
4. Ranking-by-choosing results,
5. Weak and linear orderings.

Universal properties:
Let $\mathcal{R}$ denote the set of all possible weakly complete relations definable on X.

Property 1 ($\mathcal{R}$-internal operations)**.**

1. *The* convex *combination of any finite set of such weakly complete relations remains a weakly complete relation.*
2. *The* disjunctive *combination of any finite set of such weakly complete relations remains a weakly complete relation.*

3. *The* epistemic-conjunctive *(resp.* -disjunctive*) combination of any finite set of such weakly complete relations remains a weakly complete relation.*

Examples: Concordance of linear, weak orders or semiorders, bipolar outranking (concordance-discordance) relations.

Useful properties:

We say that a binary relation $R \in \mathcal{R}$ verifies the *coduality principle* ($> \equiv \not\lesssim$), if the converse of its negation equals its asymetric part : $\min\big(r(x \,\mathrm{R}\, y), -r(y \,\mathrm{R}\, x)\big) = -r(y \,\mathrm{R}\, x)$. Let $\mathcal{R}^{cd}$ denote the set of all possible relations $R \in \mathcal{R}$ that verify the coduality principle.

Property 2 (Coduality principle)**.** *The convex and epistemic-disjunctive (resp. -conjunctive) combinations of a finite set of relations in $\mathcal{R}^{cd}$ verify again the coduality principle.*

Example: Marginal linear-, weak- and semi-orders, concordance and bipolar outranking relations, all verify the coduality principle.

2.2 The Rubis choice procedure

Pragmatic principles of the Rubis choice:

$\mathcal{P}_1$: *Elimination for well motivated reasons*:
Each eliminated alternative has to be outranked by (resp. is outranking) at least one alternative in the Rubis choice (RC).

$\mathcal{P}_2$: *Minimal size*:
The RC must be as limited in cardinality as possible.

$\mathcal{P}_3$: *Stable* and *efficient*:
The RC must not contain a self-contained sub-RC.

$\mathcal{P}_4$: *Effectively better (resp. worse)*:
The RC must not be ambiguous in the sense that it is not both a best choice as well as a worst choice recommendation.

$\mathcal{P}_5$: *Maximally significant*:
The RC is, of all potential best (resp. worst) choices, the one that is most significantly supported by the marginal "*at least as good as*" relations.

Qualifications of a choice in X

Let S be an r-valued outranking relation defined on X and let Y be a non empty subset of X, called a *choice* in X.

- Y is called *outranking* (resp. *outranked*) iff for all non retained alternative x there exists an alternative y retained such that $r(y \,\mathrm{S}\, x) > 0.0$ (resp. $r(x \,\mathrm{S}\, y) > 0.0$).
- Y is called *independent* iff for all $x \neq y$ in Y, we observe $r(x \,\mathrm{S}\, y) \leqslant 0.0$.
- Y is an *outranking kernel* (resp. *outranked kernel*) iff Y is an outranking (resp. outranked) and independent choice.
- Y is an outranking (resp. outranked) *hyper-kernel* iff Y is an outranking (resp. outranked) choice containing chordless circuits of odd order $p \geqslant 1$.

Translating the pragmatic Rubis principles in terms of choice qualifications:

$\mathcal{P}_1$: Elimination for well motivated reasons.
The RC is an *outranking choice* (resp. *outranked choice*).

$\mathcal{P}_{2+3}$: Minimal and stable choice.
The RC is a *hyper-kernel*.

$\mathcal{P}_4$: Effectivity.
The RC is a choice which is *strictly more outranking than outranked* (resp. *strictly more outranked than outranking*) .

$\mathcal{P}_5$: Maximal significance.
The RC is the *most determined* one in the set of potential outranking (resp. outranked) hyper-kernels observed in a given r-valued outranking relation.

2.3 Properties

Properties of the Rubis choice:

Property 3 (decisiveness). *Every r-valued (strict) outranking relation admits at least one outranking and one outranked hyper-kernel.*

Definition 2. Let S and S' be two r-valued outranking relations defined on X.

1. We say that S' *upgrades* action $x \in X$, denoted $S^{x\uparrow}$, if $r(x \,\mathrm{S}'\, y) \geqslant r(x \,\mathrm{S}\, y)$, and $r(y \,\mathrm{S}'\, x) \leqslant r(y \,\mathrm{S}\, x)$, and $r(y \,\mathrm{S}'\, z) = r(y \,\mathrm{S}\, z)$ for all $y, z \in X - \{x\}$.

2. We say that S' *downgrades* action $x \in X$, denoted $S^{x\downarrow}$, if $r(y\,\mathrm{S}'\,x) \geqslant r(y\,\mathrm{S}\,x)$, and $r(x\,\mathrm{S}'\,y) \leqslant r(x\,\mathrm{S}\,y)$, and $r(y\,\mathrm{S}'\,z) = r(y\,\mathrm{S}\,z)$ for all $y, z \in X - \{x\}$.

Properties of the RUBIS choice:

Let A be a subset of X. Let $RBC(S_{|A})$ (resp.$RBC(S'_{|A})$) be the RUBIS best choice wrt to S (resp. S') restricted to A and let $RWC(S_{|A})$ (resp.$RWC(S'_{|A})$) be the RUBIS worst choice wrt to S (resp. S') restricted to A.

Property 4. *The* RUBIS *choice procedure verifies following properties:*

1. $S_{|A} = S'_{|A} \;\Rightarrow\; RBC(S_{|A}) = RBC(S'_{|A})$ *(RBC local),*
2. $S_{|A} = S'_{|A} \;\Rightarrow\; RWC(S_{|A}) = RWC(S'_{|A})$ *(RWC local),*
3. $x \in RBC(S_{|A}) \;\Rightarrow\; x \in RBC(S^{x\uparrow}_{|A})$ *(RBC weakly monotonic),*
4. $x \in RWC(S_{|A}) \;\Rightarrow\; x \in RWC(S^{x\downarrow}_{|A})$ *(RWC weakly monotonic).*

It is noticeable that the RUBIS choice procedure is weakly monotonic despite the fact it does not satisfy the Super Set Property (SSP, see [1]).

3 Ranking-by-choosing

3.1 Algorithm

Let X_1 be the set X of potential decision actions we wish to rank on the basis of a given outranking relation S.

1. While the remaining set X_i ($i = 1, 2, ...$) of decision actions to be ranked is not empty, we extract from X_i the best (B_i), respectively worst (W_i), RUBIS choice recommendation and set $X_{i+1} = X_i - B_i$, respectively $X_{i+1} = X_i - W_i$.
2. Both independent iterations determine, hence, two – usually slightly different – opposite weak orderings on X: a *ranking-y-best-choosing* – and a *ranking-by-worst-choosing* order.
3. We *fuse* both rankings, the first and the converse of the second, with the help of the epistemic conjunction operator ($\oslash$) to make apparent a weakly complete ranking relation $\succsim_S$ on X. We denote $\succ_S$ the codual of $\succsim_S$.

3.2 Properties

Definition 3. We call a ranking procedure *transitive* if the ranking procedure renders a (partial) strict ordering $\succ_S$ on X with a given r-valued outranking relation S such that for all $x, y, z \in X$: $r(x \succ_S y) > 0$ and $r(y \succ_S z) > 0$ imply $r(x \succ_S z) > 0$.

Property 5. *Both the* Rubis *ranking-by-best-choosing, as well as the* Rubis *ranking-by-worst-choosing procedures, are transitive ranking procedures.*

Corollary 1. *The fusion of the ranking by* Rubis *best choice and the converse of the ranking by* Rubis *worst choice of a given r-valued outranking relation Sgives:*

1. *a weakly complete and transitive ranking of X,*

2. *which is a transitive closure of the codual of S.*

Definition 4 (Weak monotinicity)**.** We call a ranking procedure *weakly monotonic* if for all $x, y \in X$: $(x \succ_S y) \Rightarrow (x \succ_{S^{x\uparrow}} y)$ and $(y \succ_S x) \Rightarrow (y \succ_{S^{x\downarrow}} x)$,

Property 6. *The ranking by* Rubis *best choice and the ranking by* Rubis *worst choice are, both, weakly monotonic ranking procedures.*

Corollary 2. *The ranking-by-choosing, resulting from the fusion of the ranking by* Rubis *best choice and the converse of the ranking by* Rubis *worst choice, is hence a weakly monotonic procedure.*

Definition 5 (Condorcet consistency)**.** We call a ranking procedure *Condorcet-consistent* if the ranking procedure renders the same linear (resp. weak) order $\succ_S$ on X which is, the case given, modelled by the strict majority cut of the codual of a given r-valued outranking relation S.

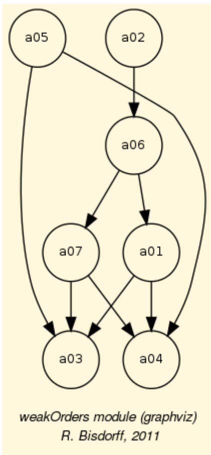
Property 7. *Both the* Rubis *ranking-by-best-choosing, as well as the* Rubis *ranking-by-worst-choosing procedures, are Condorcet consistent.*

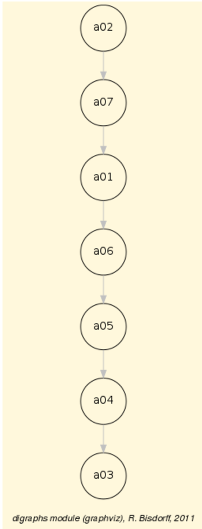
Corollary 3. *The fusion of the ranking by* Rubis *best choice and the converse of the ranking by* Rubis *worst choice of a given r-valued outranking relation S is, hence, also Condorcet consistent.*

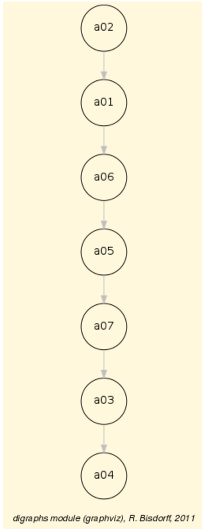
3.3 Empirical Validation

Revisiting the introductory example:

Comparing ranking-by-choosing result with Kohler's and Tideman's:



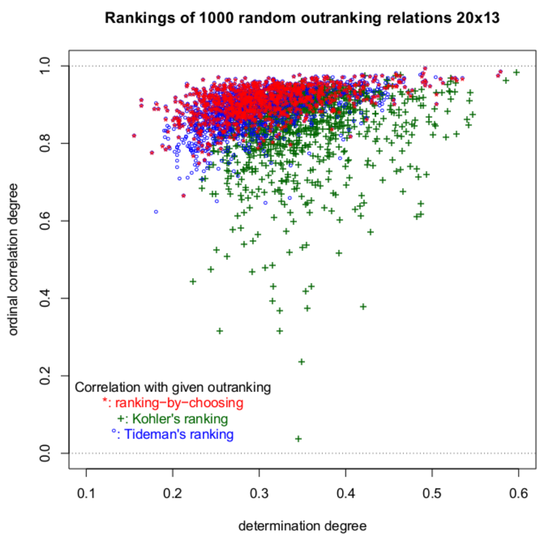




0.88% 0.90% 0.87%

Quality of ranking result:

We may compare, with a sample of 1000 random r-valued outranking relations defined on 20 actions and evaluated on 13 criteria the results obtained with respectively the RUBIS **ranking-by-choosing**, **Kohler's**, and **Tideman's** (ranked pairs) procedure.



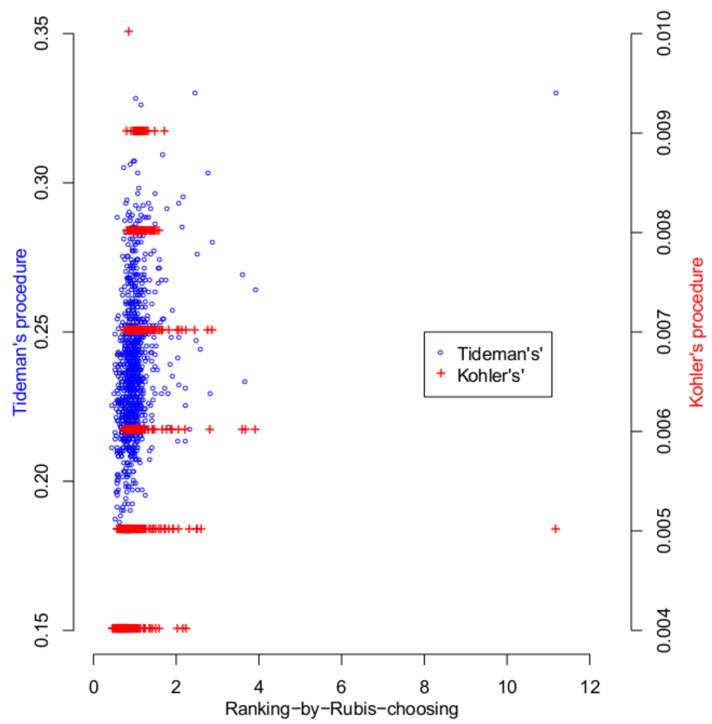
Mean extended Kendall τ correlations with r-valued outranking relation:

- *Ranking-by-choosing*: $\tau_{rbc} = +\mathbf{.906}$
- Tideman's ranking: $\tau_{tid} = +\mathbf{.875}$
- Kohler's ranking: $\tau_{koh} = +\mathbf{.835}$

r-valued determination of ranking result:

- Mean outranking significance: 0.351 (67.5% of total criteria support),
- Mean Ranking-by-choosing significance: 0.268 (63.4% of total criteria support),
- Mean covered part of significance: 0.268/0.351 = **76**%.

Scalability of ranking procedures:



Ranking execution times (in sec.) for 1000 random 20x13 outrankings:

- **Kohler**'s procedure on the *right y-axis* (less than **1/100** sec.),

- **Tideman**'s procedure on the left y-axis (less than **1/3** sec.),
- the RUBIS **ranking-by-choosing** procedure on the x-axis (mostly less than **2** sec.). But, heavy right tail (up to **11** sec. !).

Practical application

- *Spiegel (DE) On-line* 50 000 *Students' Survey* (2004) about the evaluation of 41 German universities with respect to 15 academic disciplines;
- XMCDA 2.0 encoding of performace tableau;
- Ranking-by-choosing result.

Concluding remarks

To be written ...

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